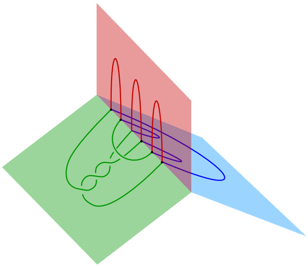

# Tangles, 2–knots, and Trisections


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## 2–knot or not 2–knot?

**Definition**

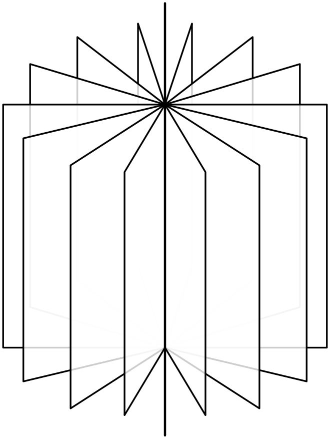
*A 2–knot is an embedded sphere in $\mathbb{R}^4$.*

1. Can a sphere even *be* knotted in $\mathbb{R}^4$?
2. What's interesting about 2–knots?
3. I like *normal* knots, how can I study 2–knots?

First, let's embed a sphere in $\mathbb{R}^3$... using cylindrical coordinates.

# A sphere in $\mathbb{R}^3$

$$\mathbb{R}^3 = \{(r, \theta, z) \mid r \geq 0, \theta \in S^1\}$$



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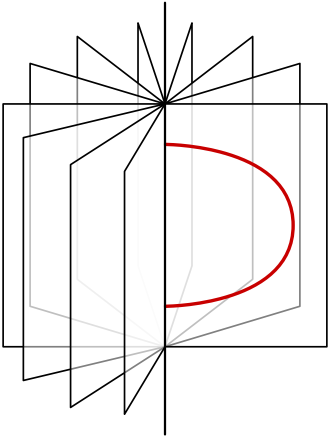

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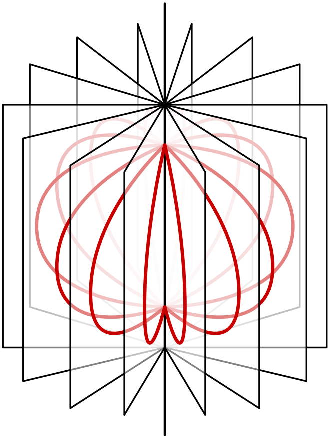

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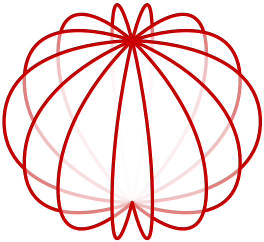

# A 2–knot in $\mathbb{R}^4$

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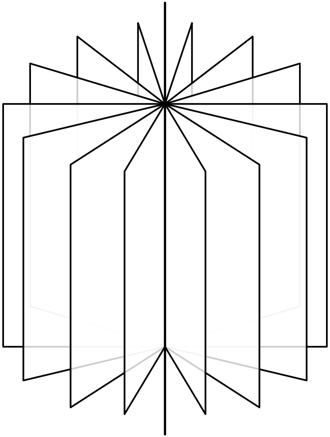

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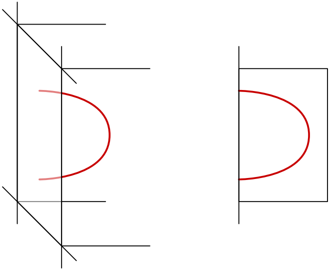

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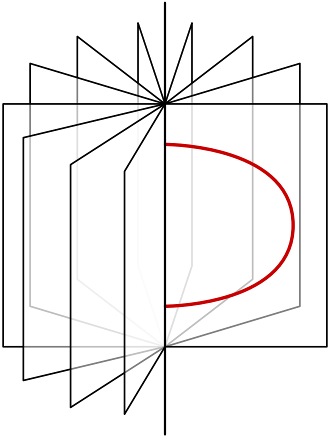

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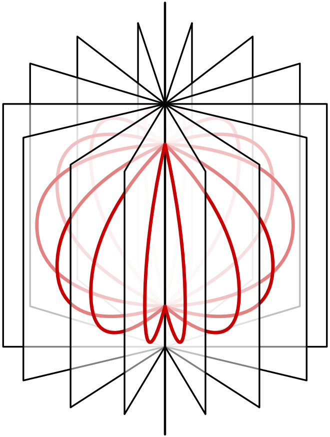

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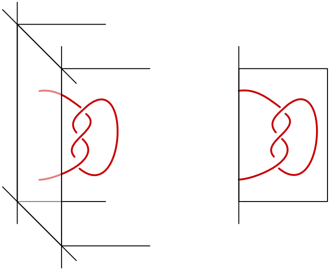

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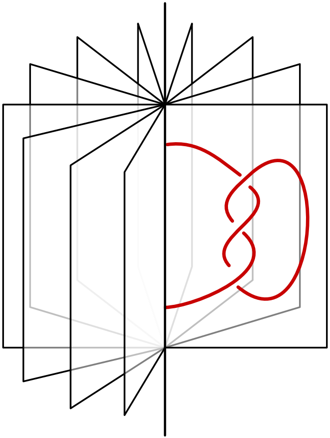

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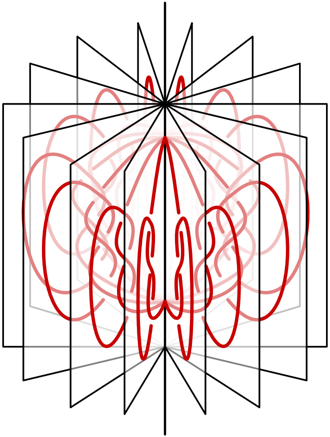

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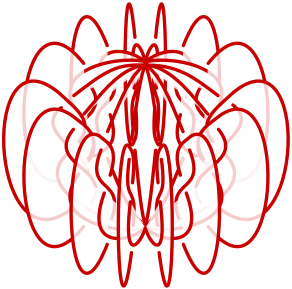


## Theorem (Artin, 1925)

*Let $K \subset \mathbb{R}^3$ be a knot in $\mathbb{R}^3$, and let $\mathcal{S}_K \subset \mathbb{R}^4$ be the spin of $K$. Then,*

$$\pi_1(\mathbb{R}^3 \setminus K) \cong \pi_1(\mathbb{R}^4 \setminus \mathcal{S}_K).$$

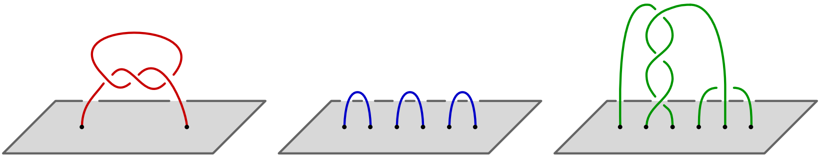
## Prime Decomposition Question

*Are there nontrivial 2–knots $\mathcal{S}_1$ and $\mathcal{S}_2$ such that $\mathcal{S}_1 \# \mathcal{S}_2$ is trivial?*

# Tangle up in blue... and red and green!

**Definition**

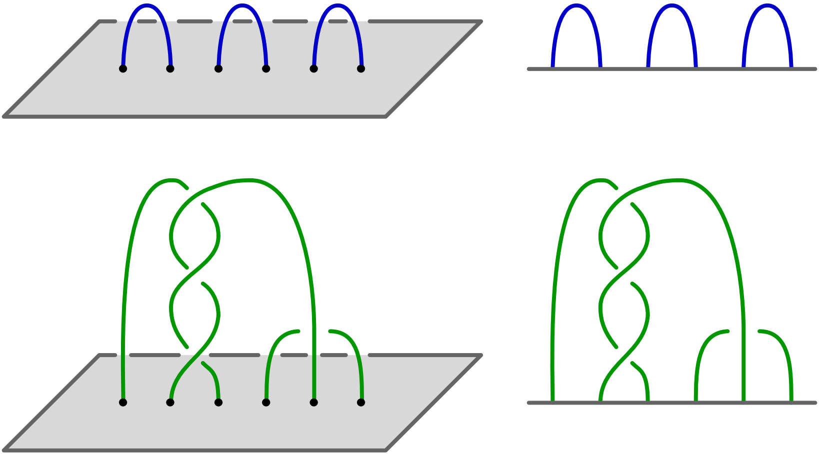
*A tangle is a collection of properly embedded arcs in* $\mathbb{R}^3_+$.



**Definition**

*A tangle is trivial if each arc has a single maximum.*

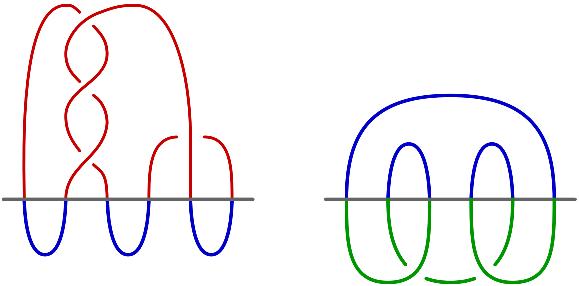
# Tangle diagrams



# From trivial tangles to knots and links

**Theorem**

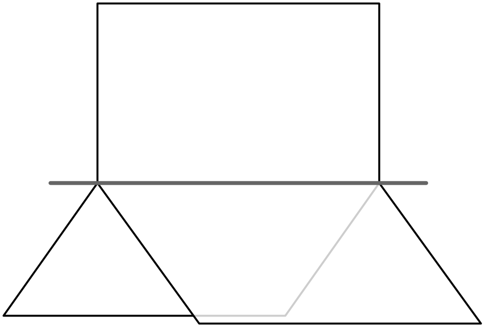
*Every link in $\mathbb{R}^3$ is the union of two trivial tangles.*



# Triplanes and spines

**Definition**

*A* *triplane* *is a choice of three half-space flanges in* $\mathbb{R}^4$.



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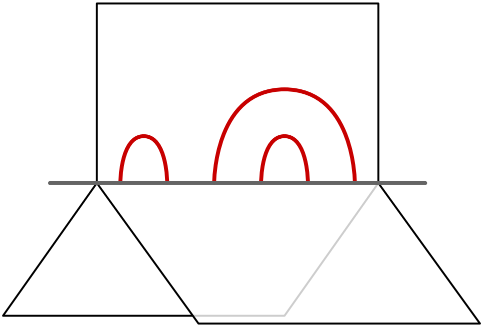

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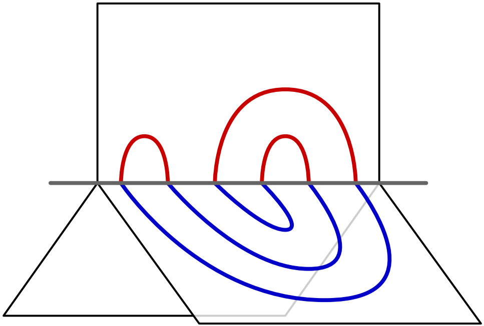

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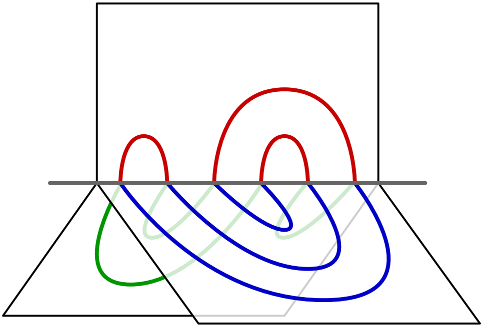

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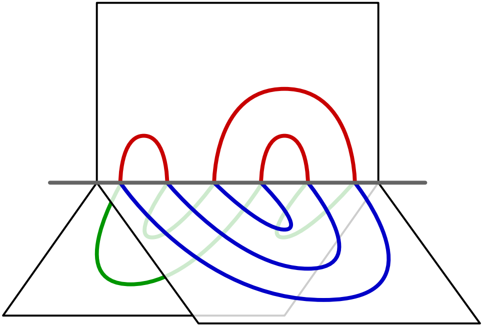


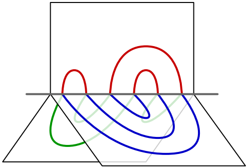
**Definition**

*A spine is a triple of trivial tangles embedded in a triplane such that each pairwise union is an unlink.*

# A spine determines a surface-knot

**Proposition**

*A spine uniquely determines a knotted surface in* $\mathbb{R}^4$.



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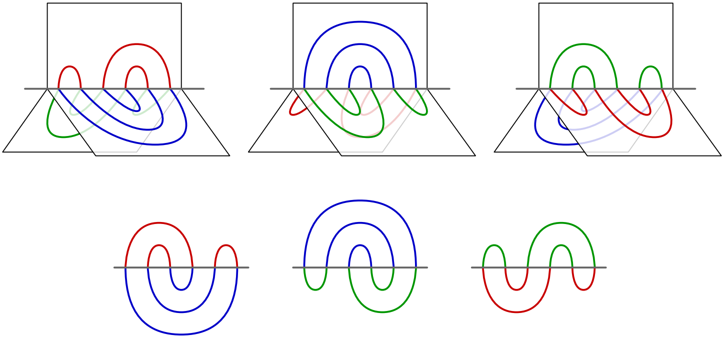

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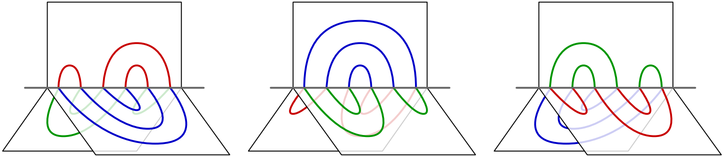
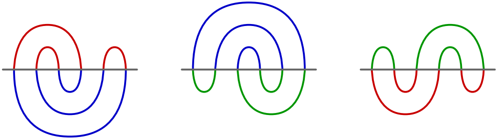




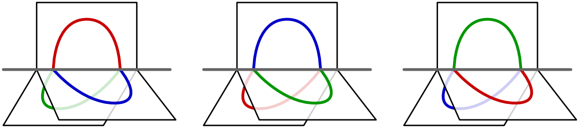
## Definition

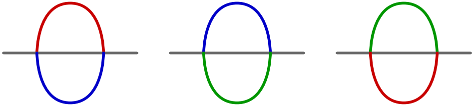
*Such a decomposition is called a (b, c)–bridge trisection, where b is the number of strands in the tangles and c is the number of components in the unlinks.*

# Examples

**Fact**

*If $\mathcal{S}$ admits a $(b, c)$–bridge trisection, then $\chi(\mathcal{S}) = 3c - b$.*





The unknotted sphere, $b = 1, c = 1$

# Examples

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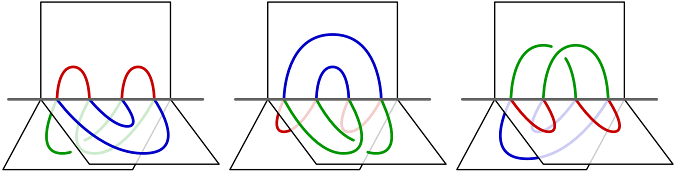

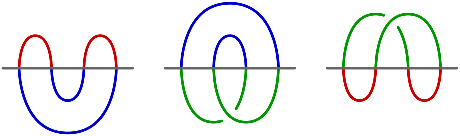

The unknotted projective plane, $b = 2, c = 1$

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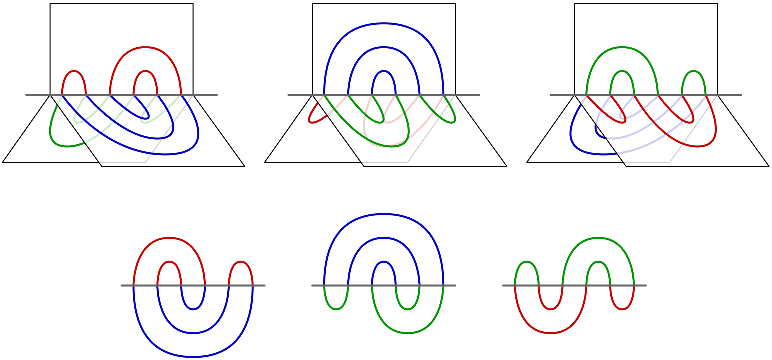
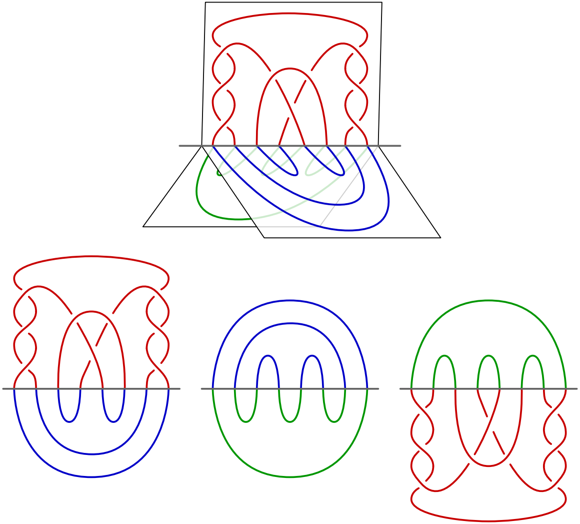


The unknotted torus, $b = 3, c = 1$

# Examples



The spin of the trefoil knot, $b = 4, c = 2$

# Main results

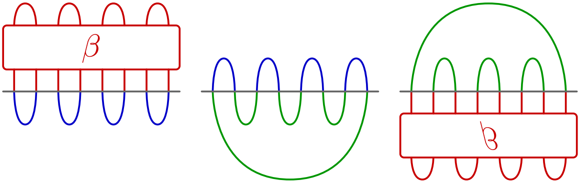
**Theorem (M.–Zupan, 2017)**

1. *Every knotted surface admits a* $(b, c)$*–bridge decomposition for some* $b \geq c \geq 1$*.*
2. *Any two bridge trisections of a fixed knotted surface can be related by a finite sequence of triplane moves.*
3. *There are only 3 irreducible* $(b, c)$*–bridge trisections with* $b \leq 3$*.*

# Open problems

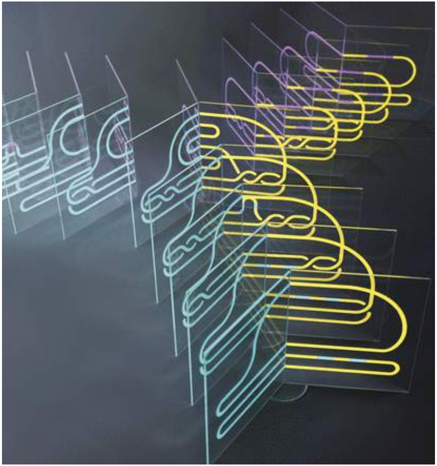
## Problem 1

*Find a nontrivial $(b,1)$–bridge trisection for $b \geq 4$.*




## Problem 2

*Show that* $b(\mathcal{S}_1 \# \mathcal{S}_2) = b(\mathcal{S}_1) + b(\mathcal{S}_2) - 1.$



Thank you!