Research Statement

My research interests lie in the field of low-dimensional topology, a major goal of which is to understand three-dimensional manifolds, four-dimensional manifolds, and the interactions between them. My research falls into three categories: trisections of four-manifolds and knotted surfaces, slice and doubly slice knots, and Dehn surgery on knots and links.

First, a major component of my current research program focusses on the trisection theory introduced in 2012 by Gay and Kirby, which represents an exciting new approach to four-dimensional topology. A trisection is a decomposition of a four-manifold into three simple pieces, and the strength of the theory is that it draws striking parallels between analogous three-dimensional techniques and machinery. Together with Alex Zupan, I have worked over the last few years to develop the theory of trisections, formalize the connections to three-dimensional topology, and extend the theory to new settings, such as the study of knotted surfaces in four-manifolds. Our group was recently awarded a highly-competitive NSF Focused Research Grant[1] to support our development of the theory of trisections. This work, which has introduced a number of interesting new research directions, is discussed in Sections 1 and 2.

Second, an important interaction between three-dimensional and four-dimensional topology arises in the study of slice knots and doubly slice knots, wherein one examines which knots arise as cross-sections of knotted and unknotted spheres in four-space. For 60 years, the study of slice knots have been central in our understanding of the connections between three-dimensional and four-dimensional topology, while the theory of doubly slice knots remains less developed. My research touches on multiple aspects of the these theories and is discussed in Sections 3, 4, and 5.

Finally, Dehn surgery is the process of removing a neighborhood of a knot before reattaching it in a novel way. This operation is robust enough that, when applied to links, encapsulates the complexity of four-dimensional topology. On the other hand, it is sensitive enough in the case of knots to encapsulate deep connections between the topology and geometry of knots and three-manifolds. Many of the most important open questions in three-dimensional topology relate to Dehn surgery on knots and links, and these questions – as discussed in Sections 1, 4, and 6 – are a recurring focus of my research.

Though these subjects represent a broad swath of low-dimensional topology, there are many natural connections between them, and my research often makes use of these connections. One theme of my research is to apply modern techniques such as Khovanov homology, Heegaard Floer homology, and trisection theory to approach classical problems in low-dimensional topology. In light of these modern machineries, the connections between these subjects offer many opportunities for future research and progress, and I have ongoing projects in all three areas, some of which is described below. Section 7 is an annotated bibliography containing brief overviews of my past results and some current projects.

1. Trisections and their classification

Trisections are an exciting new development in the theory of 4–manifolds. Introduced by Gay and Kirby in 2012 [15], they provide a 4–dimensional analogue to the concept of Heegaard splittings of 3–manifolds, and there is evidence that they will serve as an effective platform for translating well-established and effective 3–manifold techniques to the realm of 4–manifolds.

Definition 1.1. Let X be a closed, connected, orientable, smooth 4–manifold. A (g,k)*–trisection* of X is a decomposition $X = X_1 \cup X_2 \cup X_3$ such that

- $X_i \cong \natural^k(S^1 \times B^3)$ is a 4–dimensional handlebody for each i;

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- $H_{ij} = X_i \cap X_j \cong \natural^g(S^1 \times D^2)$ is a 3–dimensional handlebody for each pair i, j; and
- $\Sigma = X_1 \cap X_2 \cap X_3$ is a closed surface of genus g.

The surface Σ is called the *trisection surface*, and we say that the trisection has *genus* g.

Notice that $\partial X_i \cong \#^k(S^1 \times S^2)$ and that (Σ, H_{ij}, H_{ki}) is a genus g Heegaard splitting of ∂X_i. In fact, the trisection is completely determined by the quadruple $(\Sigma, H_{12}, H_{23}, H_{31})$ thanks to [37], which says that $\#^k(S^1 \times S^2)$ can be capped off by a 4–dimensional handlebody in a unique way. It's also worth remarking that genus g Heegaard splittings of $\#^k(S^1 \times S^2)$ are known to be standard by [68]. Thus, each pair of 3–dimensional handlebodies determines a trivial Heegaard splitting, but the combination of the three 3–dimensional handlebodies is enough to capture the complexity of the class of smooth 4–manifolds. Note that trisections are a restrictive version of Heegaard triples, which form a cornerstone within the theory of Heegaard Floer homology [54, 57, 58, 59, 60]

Theorem 1.2 (Gay-Kirby [15])**.** *Every closed, connected, orientable, smooth 4–manifold X admits a (g,k)–trisection for some $g \geq k \geq 0$, and any two trisections of X have a common stabilization.*

The only 4–manifold that admits a genus zero trisection is S^4, which can be represented as the union of three 4–balls that intersect pairwise along a 3–ball portion of their S^3 boundaries. It's an easy exercise to determine that there are only three genus one trisections, which correspond to $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$, and $S^1 \times S^3$. There is also a nice genus two trisection of $S^2 \times S^2$. Diagrams for these four standard trisections are shown in Figure 1.

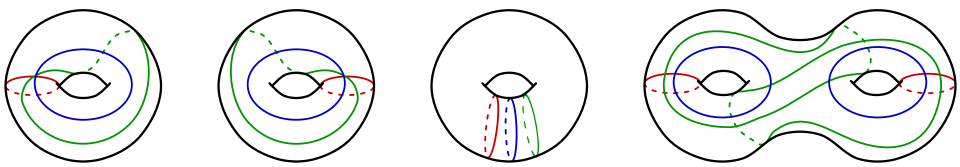


FIGURE 1. Trisection diagrams for $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$, $S^1 \times S^3$, and $S^2 \times S^2$.

Naturally, one of the first questions to arise following the introduction of trisections was to what extent one could classify trisections of low genus. To this end, Zupan and I proved the following theorem.

Theorem 1.3 (M.-Zupan [50])**.** *Every genus two trisection is equivalent to either the standard trisection of $S^2 \times S^2$ or to a connected sum of the standard genus one trisections.*

The most interesting aspect of this result is that the methods of proof are entirely 3–dimensional. We make use of a deep theorem of Homma, Ochiai, and Takahashi that states that every nontrivial genus two Heegaard diagram for S^3 contains a wave [29] and employ a complicated combinatorial argument to show that any nonstandard genus two trisection diagram can be simplified.

There is a slight generalization of trisections called *unbalanced* trisections. Here, we allow the complexity of the three pieces X_i to vary, requiring only that $X_i \cong \natural^{k_i}(S^1 \times B^3)$ for some $k_i \geq 0$. Such a decomposition is called a $(g; k_1, k_2, k_3)$*–trisection.* For example, there are three unbalanced genus one trisections.

Theorem 1.4 (M.-Schirmer-Zupan [45])**.** *Every $(g; k_1, k_2, k_3)$–trisection with $k_1 \geq g-1$ is equivalent to a connected sum of (balanced or unbalanced) genus one trisections.*

Again, the most interesting aspect of the theorem is the way in which 3–manifold techniques are used to prove a 4–dimensional result. In this case, we make use of the 3–fold symmetry inherent to trisections, called *handle triality.* This idea is analogous to the classical approach of analyzing the dual description of a manifold by turning it upside down. Handle triality puts 1–, 2–, and 3–handles on equal footing and allows for a 4–manifold with a trisection to be analyzed in *six* different ways.

For example, a $(g; 0, g-1, 1)$–trisection gives rise to a Dehn surgery on knot in S^3 that yields $S^1 \times S^2$. Such a surgery is trivial, by Gabai [14]. On the other hand, by permuting the parameters, we get a $(g; 0, 1, g-1)$–trisection, which gives rise to a c–component link in S^3 with a surgery to $\#^c(S^1 \times S^2)$. Links with such surgeries are the subject of the Generalized Property R Conjecture (see [19]). Here we state a stable version that is most appropriate in this setting.

Generalized Property R Conjecture. *Suppose that L is a c–component link in S^3 with a surgery to $\#^c(S^1 \times S^2)$. Let L' be the split union of L with an r–component unlink for some $r \geq 0$. Then there exists a sequence of handleslides converting L' into a $(c+r)$–component unlink.*

Thus, by permuting the order of the parameters, we can re-phrase trisection problems in terms of different types of 3–manifold questions, some that have been solved and some that have not. A good example of this is the following corollary.

Corollary 1.5 (M.-Schirmer-Zupan [45])**.** *Let L be a c–component link in S^3 with an integral Dehn surgery to $\#^c(S^1 \times S^2)$. If L has tunnel number at most c, then L satisfies the Generalized Property R Conjecture.*

Gompf, Scharlemann, and Thompson [19] have produced potential counterexamples to this conjecture and shown that if their links satisfy the conjecture then the presentations P_n of the trivial group given by

$$\langle x, y \,|\, xyx = yxy, x^n = y^{n+1} \rangle$$

are Andrews-Curtis trivial [1], which is widely believed to be false for $n \geq 3$. Each of the examples of Gompf, Scharlemann, and Thompson gives rise to an interesting trisection of S^4 [46, 49]. Thus, we have the following important connection between trisections and the Andrews-Curtis Conjecture.

Corollary 1.6. *If every nontrivial trisection of S^4 is stabilized, then the presentations P_n are Andrews-Curtis trivializable.*

The question of whether or not every nontrivial trisection of S^4 is stabilized is a 4–dimensional version of Waldhausen's Theorem for 3–manifolds [68]. In [49] and in further work in progress [46], Zupan and I give potential counter-examples to this conjecture, namely, trisections of S^4 that are likely not stabilized. One facet of this work is the connection to fibered homotopy-ribbon knots discussed at the end of Section 5 below.

2. Bridge trisections of knotted surfaces

A *trivial b–strand tangle* is a pair (B, α) where B is a 3–ball and $\alpha \subset B$ is a collection of b properly embedded, disjoint arcs that can be simultaneously isotoped to lie in ∂B. A *trivial c–disk system* is a pair $(X, \mathcal{D})$ where X is a 4–ball and $\mathcal{D}$ is a collection of c properly embedded, disjoint disks that can be simultaneously isotoped rel–∂ to lie in ∂X.

Definition 2.1. Let $\mathcal{S}$ be a closed surface in S^4. A (b, c)*–bridge trisection* of $(S^4, \mathcal{S})$ is a decomposition $(S^4, \mathcal{S}) = (X_1, \mathcal{D}_1) \cup (X_2, \mathcal{D}_2) \cup (X_2, \mathcal{D}_2)$ such that

- $S^4 = X_1 \cup X_2 \cup X_3$ is the genus zero trisection of S^4;
- $(X_i, \mathcal{D}_i)$ is a trivial c–disk system for each i; and
- $(B_{ij}, \alpha_{ij}) = (X_i, \mathcal{D}_i) \cap (X_j, \mathcal{D}_j)$ is a trivial b–strand tangle for each pair i, j.

The surface $\Sigma = X_1 \cap X_2 \cap X_3$ is called the *bridge surface*.

Notice that $(S^3, L_i) = \partial(X_i, \mathcal{D}_i)$ is a c–component unlink, and that $(S^3, L_i) = (B_{ij}, \alpha_{ij}) \cup (B_{ki}, \alpha_{ki})$ is a b–bridge splitting of L_i. Any two trivial c–disk systems are isotopic rel–∂ [31, 39], so it follows that the bridge trisection is completely determined by the triple of trivial tangles (B_{ij}, α_{ij}).

Theorem 2.2 (M.-Zupan [48])**.** *Let $\mathcal{S}$ be a closed surface in S^4. Then $\mathcal{S}$ admits a (b, c)–bridge trisection for some $b \geq c \geq 1$, and any two bridge trisections of $\mathcal{S}$ are stably equivalent.*

Note that $\mathcal{S}$ is not required to be connected nor orientable. Zupan and I have recently extended the theory of bridge trisections to surfaces in any closed 4–manifold [47]. This has produced a very picture in the special case of complex curves in simple complex surfaces. For now, we return to the interesting special case of surfaces in S^4.

A bridge trisection can be represented diagrammatically by projecting the strands of the three trivial tangles onto the union of the equatorial disks of the tangles. The resulting collection of tangle diagrams is called a *tri-plane diagram*: a triple of trivial b–strand tangle diagrams whose pairwise unions are unlinks of c–components. See Figure 2.

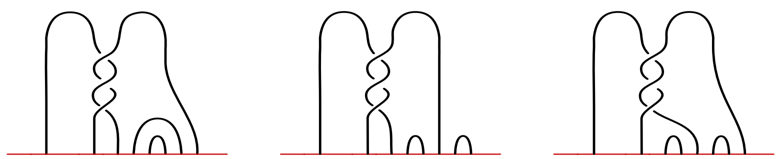


Figure 2. A $(4, 2)$–tri-plane diagram for the spun trefoil.

Corollary 2.3. *Every closed surface $\mathcal{S}$ in S^4 can be represented by a (b, c)–tri-plane diagram for some $b \geq c \geq 1$, and any two diagrams for $\mathcal{S}$ are related by a finite sequence of tri-plane moves.*

Bridge trisections are closely related to trisections of 4–manifolds. Let $\mathcal{B}$ be a (b, c)–bridge trisection of a surface $\mathcal{S}$ in S^4, and let $X_\mathcal{S}$ denote the double cover of S^4, branched along $\mathcal{S}$. Then $X_\mathcal{S}$ admits a $(b-1, c-1)$–trisection whose pieces are simply the branched double covers of the pieces of $\mathcal{B}$. By [22], every genus two handlebody is the branched double cover of a trivial 3–strand tangle, so we get the following corollary to Theorem 1.3.

Corollary 2.4. *Every (b, c)–bridge trisection with $b \leq 3$ is standard, and the surface is unknotted.*

In [48], we also construct knotted surfaces with minimal bridge trisections by spinning torus knots, giving infinitely many distinct 2–knots with minimal $(3p-2, p)$–bridge trisections for each $p \geq 2$.

Interestingly, all known examples of 2–knots with 4–bridge trisections are unions of ribbon disks for 2–bridge knots, so we are led to the following question.

Problem 2.5. *Classify 4–bridge trisections of 2–knots.*

Another interesting class of surfaces are those that are orientable and admit $(b, 1)$–bridge trisections. By topological surgery theory [12, 13, 26, 32, 34], such surfaces are topologically unknotted, since their complements have infinite cyclic fundamental group. The Unknotting Conjecture [33] states that any such surface is smoothly unknotted.

Question 2.6. *Is there a smoothly knotted surface whose complement has infinite cyclic fundamental group? Can such a surface admit a $(b, 1)$–bridge trisection for some b?*

In general, it is not known whether knotted surfaces admit prime decompositions – even in the simple case of knotted spheres. In fact, it is known that there are 2–knots such that the connected sum with an unknotted projective plane gives the unknotted projective plane [67]. Even more intriguingly, we have the following.

Kinoshita Conjecture. *Every knotted projective plane is the connected sum of a 2–knot and an unknotted projective plane.*

This question was the subject of a 2016 REU project with James Dix, who is an undergraduate at The University of Texas at Austin. Together, we have recently produced Klein bottles that have meridians of order four. Hence, they cannot contain an unknotted projective plane or Klein bottle summand, which answers a Klein bottle version of the Kinoshita Conjecture in the negative.

One of the most important reasons for studying knotted surfaces in four-space is to understand the four-manifolds obtained from various surgery operations on these knotted surfaces. For example, the *Gluck twist* is the operation given by removing a neighborhood of a 2–knot and re-gluing using the unique non-trivial diffeomorphism of the boundary. The result of a Gluck twist on a 2–knot is easily seen to be a homotopy four-sphere, but it is unknown if such manifolds are diffeomorphic to S^4 [18]. This operation is the subject of joint work with David Gay, where our goal is to give a trisection perspective. In particular, we have proved the following.

Theorem 2.7. *If a 2–knot $\mathcal{S}$ admits a (b,c)–bridge trisection, then the result of a Gluck twist on $\mathcal{S}$ admits a (g,k)–trisection with $g = b$ and $k = c$. Moreover, the corresponding diagrams are related in a simple way.*

3. Doubly slice knots

In 1962, Fox included the following question in his list of problems in knot theory [11].

Question 3.1. *Which knots can appear as the cross-section of the unknotted S^2 in S^4?*

Since then, such knots have come to be called *doubly slice*. If one allows the S^2 to be knotted in S^4, then the cross-section is simply called *slice*. Over the last 50 years, many advances in low-dimensional topology have led to breakthroughs in the study of slice knots, while the study of doubly slice knots has remained must less well-explored. In fact, there is the following technical obstruction to studying doubly slice knots in direct analogy with slice knots.

Question 3.2. *Do there exist knots K and J such that K and $K\#J$ are doubly slice, but J is not?*

With Question 3.2 in mind, we say that K_0 and K_1 are *smoothly doubly concordant* if there exist smoothly doubly slice knots J_0 and J_1 such that $K_0\#J_0$ and $K_1\#J_1$ are ambient isotopic. Under connected sum, the collection of knots in S^3 modulo smooth double concordance forms an abelian group, denoted $\mathcal{C}_\mathcal{D}$, called the *smooth double concordance group*. Analogously, we can form the *topological* and *algebraic double concordance groups*, $\mathcal{C}_\mathcal{D}^{top}$ and $\mathcal{G}_\mathcal{D}$. These groups can be organized nicely via natural surjections:

$$\mathcal{C}_\mathcal{D} \xrightarrow{\Psi} \mathcal{C}_\mathcal{D}^{top} \xrightarrow{\Phi} \mathcal{G}_\mathcal{D}.$$

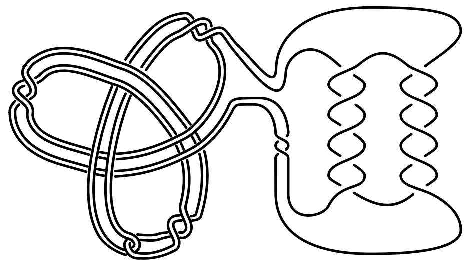
In 1983, Gilmer and Livingston gave the first examples of slice knots that are algebraically doubly slice, but not doubly slice [17], and more recently, Kim [35] extended the techniques of Cochran, Orr, and Teichner to show that the collection of such knots has a rich filtered structure.

The seminal work of Freedman [12, 13] and Donaldson [9] in the early 1980s, illuminated the difference between the smooth and topological categories in dimension four. Immediately, these differences were detected in the study of slice knots [6], and the blossoming of knot Floer homology [55, 56, 61, 62] has recently led to an improved understanding of this difference (e.g., [24, 25, 28, 53]).

For many years, nothing was known about the distinction between the smooth and topologically locally flat categories for doubly slice knots. However, in [42], I was able to prove theorems illustrating this distinction.

Theorem 3.3 (M. [42]). *There exists an infinite family of smoothly slice knots that are topologically doubly slice but not smoothly doubly slice.*

These knots are formed by infecting smoothly doubly slice knots with Whitehead doubles of the trefoil.



A geometric argument shows the resulting knots are still topologically doubly slice (and smoothly slice), and a calculation of the Heegaard Floer homology correction terms [54] using the surgery exact sequence [61, 62] shows that the double branched covers cannot embed smoothly in S^4. Thus, the knots cannot be smoothly doubly slice.

The proof has the following interesting corollary.

Corollary 3.4. *There exists an infinite family of rational homology 3–spheres that embed in S^4 topologically but not smoothly.*

All previously known examples of such objects were integer homology spheres [12, 63] and didn't bound smooth rational homology four-balls; the present examples have nontrivial first homology and bound smooth rational homology balls. This points to a 3–manifold version of Question 3.2.

Question 3.5. *Do there exist closed 3–manifolds M and N such that M and $M\#N$ embed smoothly in S^4, but N does not?*

Though Question 3.2 remains open, my work shows that the correction terms actually provide obstructions to a knot being *stably* doubly slice, which allows for a stronger restatement of Theorem 3.3.

Theorem 3.6 (M. [42]). *There is an infinitely generated subgroup $\mathcal{S}$ inside $\ker(\Psi)$ that is generated by smoothly slice knots whose order in $\mathcal{C}_\mathcal{D}$ is at least three.*

It seems reasonable to conjecture that $\mathcal{S}$ describes a free abelian direct summand of $\mathcal{C}_\mathcal{D}$. Nothing else is known about the structure of the kernels of Ψ and Φ, and a primary goal of my future research is to explore these objects further. Along these lines, we can ask the following question.

Question 3.7. *What is the structure of $\ker(\Phi)$ and $\ker(\Psi)$?*

Other than what has been described above, very little is known about these subgroups. It would be nice to find elements of finite order in either kernel. Additionally, a conjecture of Gordon (see [23]) states that every element of order 2 in $\mathcal{C}$ is represented by a negative-amphicheiral knot. It is interesting to wonder if a similar statement holds for elements of $\mathcal{C}_\mathcal{D}$.

A natural first step to trying to answer these important questions is to gain a firm understanding of basic examples of doubly slice knots. To this end, Chuck Livingston and I gave an analysis of doubly slice knots up to twelve crossings [40], building on Sumners' earlier work [64].

Theorem 3.8 (Livingston-M. [40]). *Among all knots with at most 12 crossings, all but three knots can be shown to be either smoothly doubly slice or not topologically doubly slice, and only the Conway knot, 10_{n34}, is known to be topologically doubly slice but not smoothly doubly slice.*

Theorem 3.8 leaves open four cases for future study, and gives motivation to expand the classification beyond twelve crossings. The techniques involved range from classical algebraic techniques to modern topological techniques involving twisted Alexander polynomials. Interestingly, smooth invariants failed to provide assistance for these low-crossing knots but will likely be useful in the future.

4. Cabled slice knots and reducible surgeries

Dehn surgery is one of the simplest and most important operations in 3–manifold topology. In short, Dehn surgery is the process of removing a solid torus neighborhood of a knot and gluing it back in with a different identification, called a slope. One of the most basic open questions asks when the result of Dehn surgery can be a reducible manifold (a manifold containing an essential two-sphere). Let $J_{p,q}$ denote the (p,q)–cable of J, and let $S^3_r(K)$ denote the result of performing Dehn surgery on K along slope $r \in \mathbb{Q}$. The Cabling Conjecture [20] asserts that every reducible Dehn surgery is pq–surgery on a (p,q)–cabled knot. A more accessible version of this very hard conjecture is the following.

Two Summands Conjecture. *If K is a nontrivial knot in S^3 and $S^3_r(K)$ is reducible, then $S^3_r(K) \cong Y_1 \# Y_2$, with Y_1 and Y_2 irreducible.*

The Two-Summands Conjecture is true for knots with bridge number at most five and positive braid closures [74]. In [43], I verified the Two Summands Conjecture for slice knots.

Theorem 4.1 (M. [43])**.** *A slice knot in the 3–sphere cannot admit a reducible surgery with three irreducible summands.*

In fact, Theorem 4.1 holds for any knot K with $V_0(K) = V_0(\overline{K}) = 0$, where $V_0(K)$ is a Heegaard Floer theoretic knot invariant coming from the knot Floer complex that determines the correction terms of surgeries on K [61, 62], and $\overline{K}$ denotes the mirror of K. The condition that $V_0(K) = 0$ suffices in the case that the surgery is positive. Along these same lines, we have the following observation.

Theorem 4.2 (M. [43])**.** *Let $K = J_{p,q}$ with $p, q > 0$. If K is slice, then $V_0(J) = 0$.*

A key insight in the proof is the following fact, which can be proved using classical methods: If $J_{p,q}$ is algebraically slice, then $q = 1$. This leads to the following natural conjecture.

Conjecture 4.3. *The cabled knot $J_{p,q}$ is slice if and only if J is slice and $q = 1$.*

Since any $(p,1)$–cable of slice knot is also slice, one direction of the conjecture is true. Theorem 4.2, along with [27] and [73], give evidence that the conjecture is true at the level of Heegaard Floer homology. Homology invariants of cabled knots have also been studied recently in [4, 10]. Note that Conjecture 4.3 is true for fibered knots in the homotopy-ribbon setting by Theorem 8.5 of [51].

5. Fibered ribbon disks

One of the most important open questions in low-dimensional topology is the Slice-Ribbon Conjecture, which asserts that every slice knot is ribbon. A knot is *ribbon* if it bounds a disk in B^4 that can be built with only 0–handles and 1–handles. A knot K is called *homotopy-ribbon* if it has a slice disk D such that there exists a surjection from $\pi_1(S^3 \setminus K)$ onto $\pi_1(B^4 \setminus D)$.

Celebrated results of Casson and Gordon [3] and Cochran [5] give characterizations of fibered homotopy-ribbon knots in dimensions three and four, respectively. Larson and I interpolated these results in the following theorem. A *disk knot* is a properly embedded two-disk in the 4–ball, considered up to ambient isotopy. A disk knot D is called *homotopy-ribbon* if it is a homotopy-ribbon disk for ∂D.

Theorem 5.1 (Larson-M. [36])**.** *Let D be a fibered disk knot in B^4 with fiber H. Then D is homotopy-ribbon if and only if H is homeomorphic to a handlebody.*

We also describe a twisting operation for handlebody bundles that can be used to produce new fibered disk knots from a given one. This leads to infinite families of distinct fibered ribbon knots with homotopy-equivalent exteriors. In general, disk knots constructed in this way are not obviously ribbon; thus, we can build infinite families of potential counter-examples to the Slice-Ribbon Conjecture.

Given a 2–knot $\mathcal{S} \subset S^4$, we call 1–knot K a *symmetric equator* of $\mathcal{S}$ if $\mathcal{S}$ is the double of a disk along K (in some contractible 4–manifold). Larson and I show that every fibered homotopy-ribbon 2–knot has infinitely many symmetric equators [36]. This leads to interesting questions about the interplay between fibered ribbon 1–knots and fibered ribbon 2–knots, such as the following.

Question 5.2. *Which fibered ribbon 1–knots occur as a symmetric equator of spun fibered knots?*

In [49] and in further work in preparation, Zupan and I have shown how every fibered, homotopy-ribbon knot of genus g gives rise to a $(2g; g, g, 0)$–trisection of a homotopy four-sphere. This generalizes the examples of [19], and each such trisection has relevance to Generalized Property R Conjecture and the Andrews-Curtis Conjecture for the corresponding group presentation. In each specific case, these conjectures hold if the given trisection becomes standard after stabilizing in only two of the three directions.

6. Exceptional Seifert fibered surgery

Around 1960, Lickorish and Wallace proved that every 3–manifold can be obtained by Dehn surgery on some link in S^3 [38, 69]. Since then, a major problem in low-dimensional topology has been to understand the special case of Dehn surgery on knots in S^3. By Thurston's Geometrization for knot complements [66], we know that every knot is either a torus knot, a satellite knot, or a hyperbolic knot. Surgeries on torus knots are well-understood [52], as are surgeries on satellite knots [21]. Moreover, Thurston showed that a hyperbolic knot can only have finitely many surgeries that produce non-hyperbolic 3–manifolds [65]. Such surgeries are called *exceptional*, and have been the subject of decades of study.

Exceptional surgeries can be divided into three categories: reducible, toroidal, and Seifert fibered. While open questions remain in all cases, understanding the case of Seifert fibered surgeries has presented a particularly formidable challenge over the years, and very little is known or even conjectured about such surgeries. Seifert fibered spaces split naturally into three cases: lens spaces, toroidal Seifert fibered spaces, and small Seifert fibered spaces.

It is natural to ask which hyperbolic knots admit Seifert fibered surgeries. Building on work of Wu and Brittenham-Wu [2, 70, 71, 72], I was able to give a classification of small Seifert fibered surgeries on hyperbolic pretzel knots, as well as a near-classification for Montesinos knots. Together with [30], this gives a complete classification for a large class of knots called arborescent knots.

Theorem 6.1 (M. [41])**.** *There is a complete classification of small Seifert fibered surgeries on hyperbolic arborescent knots.*

Work in progress with Brandy Doleshal involves a different approach. Building on Dean's influential thesis [7], we produce large classes of knots with common Seifert fibered surgeries. Our goal is to understand when these knots have the same Seifert fibered surgery, to find knots with multiple Seifert fibered surgeries, and to search for a positive answer to the following question.

Question 6.2. *Can a toroidal Seifert fibered space be realized as Dehn surgery on a hyperbolic knot?*

One of the most intriguing (and difficult) conjectures about exceptional Dehn surgery asserts that every Seifert fibered surgery is integral. My work verifies this conjecture for arborescent knots.

Corollary 6.3. *Every Seifert fibered surgery on a hyperbolic arborescent knot is integral.*

7. Annotated Bibliography

In *Genus two trisections are standard* [50], Alexander Zupan and I classify which 4–manifolds admit genus two trisections. The proof is entirely 3–dimensional, giving evidence that the theory of trisections will be able to successfully promote 3–dimensional techniques to solve 4–dimensional problems.

In *Classification of trisections and the Generalized Property R Conjecture* [45], Trent Schirmer, Zupan, and I give classification results for a family of unbalanced trisections. One corollary to this work is a relationship between the Generalized Property R Conjecture (and, hence, the Andrews-Curtis Conjecture) and the problem of simplifying trisections of S^4.

In *Bridge trisections of knotted surfaces in* S^4 [48], Zupan and I adapt the theory of trisections to the setting of knotted surfaces in S^4. The resulting decompositions, called bridge trisections, are analogous to bridge splittings of knots and links in S^3, and offer a new measure of complexity for knotted surfaces, as well as a new diagrammatic theory for knotted surfaces.

In *Bridge trisections of knotted surfaces in four-manifolds* [47], Zupan and I extend the theory of bridge trisections to arbitrary four-manifolds, giving a diagrammatic approach to this broad area of four-dimensional topology. We find that complex curves in $\mathbb{CP}^2$ admit efficient bridge trisections.

In *Characterizing Dehn surgery on links via trisections* [49], Zupan and I formalize the connection between Dehn surgery and trisections and give a program to disprove the Generalized Property R Conj.

In *Trisections and spun 4–manifolds* [44], I give a rich list of examples of minimal trisections, pushing the connection between three-dimensional and four-dimensional techniques.

In *Distinguishing topologically and smoothly doubly slice knots* [42], I use the Heegaard Floer homology correction terms to exhibit an infinite family of smoothly slice knots that are topologically doubly slice but not smoothly doubly slice. One corollary is that there exists an infinite family of rational homology spheres that embed in S^4 topologically but not smoothly.

In *Doubly slice knots with low crossing number* [40], Chuck Livingston and I attempt to catalogue all doubly slice knots up to 12 crossings. We were able to exhibit 18 new smoothly doubly slice knots, and obstruct all but four of the remaining knots from being topologically doubly slice.

In *A note on cabled slice knots and reducible surgeries* [43], I show that a (p, q)–cable of a knot is algebraically slice, then $q = 1$ and that if it is slice, then a certain Heegaard Floer invariant must vanish for the companion knot. I also show that Dehn surgery on a slice knot cannot produce a 3–manifold with three irreducible summands.

In *Fibered ribbon disks* [36], Kyle Larson and I give a characterization of fibered homotopy-ribbon disks as those whose fiber is a handlebody and introduce an analogue of the Stallings twist for such fibrations, which we interpreted in terms of 4–dimensional surgery operations. Our twisting operation can be used to produce potential counterexamples to the Slice-Ribbon Conjecture.

In *Small Seifert fibered surgery on hyperbolic pretzel knots* [41], I classify Seifert fibered surgeries on hyperbolic Montesinos knots. As a corollary, I verify that Montesinos knots satisfy the Seifert Fibered Space Conjecture, which posits that all exceptional Seifert fibered space surgeries are integral.

In *Trisections and surgery operations on 2–knots* [16]*, David Gay and I give a method for trisecting the complement of a 2–knot (in any four-manifold) and show how one can obtain trisection diagrams for the four-manifolds obtained via the Gluck twist and surgery operations.

In *Fibered homotopy-ribbon knots and Generalized Property R* [46]*, Zupan and I show that every fibered homotopy-ribbon knot gives rise to (often infinitely many) potential counter-examples to the Generalized Property R Conjecture by producing, in each case, a trisection of a homotopy four-sphere.

In *Irreducible knotted Klein bottles* [8]*, James Dix and I give a new construction of knotted Klein bottles in S^4 and produce examples of knotted Klein bottles whose meridians have order four in the fundamental group of the complement.

* *Articles in preparation*

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