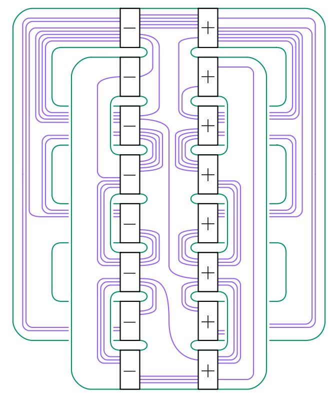
My research interests lie in the field of low-dimensional topology, a major goal of which is to understand three-dimensional spaces, four-dimensional spaces, and the interactions between them. An important facet of this field is the study of knotted curves in dimension three and knotted surfaces in dimension four. My research program spans a diverse array topics and problems in low-dimensional topology that can be roughly organized into the following five inter-related and overlapping themes, which I briefly overview before giving more detailed accounts in the sections that follow. Section 8 of this research statement is a briefly annotated bibliography of my work.

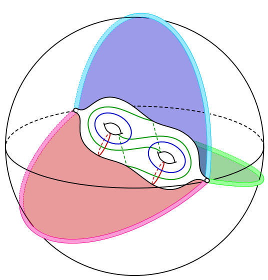
New approaches to understanding the four-sphere

The most important unsolved problem in low-dimensional topology is the Poincaré Conjecture, which asserts that every four-manifold with the homotopy type of the four-sphere is diffeomorphic to the four-sphere. Historically, potential counter-examples have been produced, then standardized. In recent work, Alex Zupan and I have shed new light on this conjecture, showing that the conjecture holds when the manifold admits a simple handle-structure and describing large families of new potential counter-examples, one of which is described by the two-component link shown on the right. Our approach adapts the theory of thin position for Heegaard splittings to the study of fibered, ribbon knots and is described in Section 1. This work also relates to other work with Zupan where we give a program to disprove the Generalized Property R Conjecture, which also relates to understanding handle-structures on homotopy four-spheres. This work is discussed in Section 7.



Developing the Theory of Trisections

A major component of my current research program focusses on the theory of trisections introduced in 2012 by Gay and Kirby, which represents an exciting new approach to four-dimensional topology.



A trisection is a decomposition of a four-manifold into three simple pieces, as depicted by the schematic to the left. One strength of the theory is that it draws striking parallels between analogous three-dimensional techniques and machinery including Heegaard theory and Dehn surgery. Zupan and I have worked over the last few years to develop the theory of trisections, formalize the connections to three-dimensional topology, and extend the theory to new settings, such as the study of knotted surfaces in four-manifolds. Our group was recently awarded a highly-competitive National Science Foundation Focused Research Grant[1] to support our development of the theory. This work, which has introduced a number of interesting new research directions, is discussed in Sections 2 and 3. For example, one of the most intriguing open questions in trisection theory asks whether trisection genus is additive. An affirmative answer to this question would imply a number of conjectures about four-manifolds, including the Poincaré Conjecture.

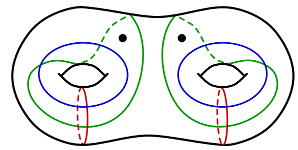
Studying knotted surfaces using bridge trisections

An important part of understanding smooth four-manifolds involves studying embedded surfaces, which can be knotted in this dimension. There are many aspects to the study of knotted surfaces. For

[1]DMS-1758087 – *FRG: Collaborative Research: Trisections – New Directions in Low-Dimensional Topology*

example, the study of knotted spheres in four-space is the natural analog of the the more familiar (three-dimensional) knot theory and represents a classical area of study in which many basic problems remain unsolved. In another vein, the minimal genus among surfaces representing a fixed second homology class carries significant information about the smooth topology of the ambient four-manifold. Lastly, we have the study of complex curves in complex surfaces, a vastly important area that touches many disciplines across mathematics.

I have a great interest in each of these (as well as other) aspects of knotted surface theory, and, to this end, Zupan and I have developed the notion of a bridge trisection. The theory of bridge trisections allows for several important developments in knotted surface theory, which are discussed in detail in Section 3.

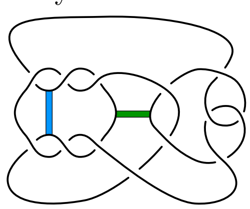


First, it gives a diagrammatic calculus for studying knotted surfaces in four-space and, more generally, the diagrammatic representation of surfaces in arbitrary four-manifolds. For example, every knotted sphere can be represented by placing two points in a trisection diagram for a four-manifold, as shown to the right. Second, it gives rise to an interesting interplay between the complexity of a surface and that of its ambient four-manifold. Peter Lambert-Cole and I have shown that for many classes of complex curves and complex surfaces, these complexities are as simple as possible. Lastly, Dave Gay and I have shown how trisections and bridge trisections can be used to study various surgery operations on knotted spheres, such as the Gluck twist.

The development of the theory of bridge trisections and its connection to symplectic geometry will be the subject of an upcoming six-person American Institute of Mathematics SQuaRE Grant.

Slice and doubly slice knots

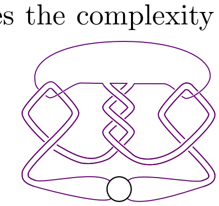
An important interaction between three-dimensional and four-dimensional topology arises in the study of slice knots and doubly slice knots, wherein one examines which knots arise as cross-sections



of knotted and unknotted spheres in four-space. For 60 years, the study of slice knots have been central in our understanding of the connections between three-dimensional and four-dimensional topology, while the theory of doubly slice knots remains less developed. One corollary to my work in this area is that there are infinite families of rational homology three-spheres that bound smooth rational homology four-balls and that admit a locally flat embedding into four-space, but no such embedding that is smooth. A necessary part of this portion of my research program, which is discussed in Sections 4 and 5, is the use classical and modern invariants such as twisted Alexander polynomial, Khovanov theoretic, and Heegaard Floer theoretic invariants.

Dehn surgery on knots and links

Dehn surgery is the process of removing a neighborhood of a knot before reattaching it in a novel way. This operation is robust enough that, when applied to links, encapsulates the complexity of four-dimensional topology. On the other hand, it is sensitive enough in the case of knots to encapsulate deep connections between the topology and geometry of knots and three-manifolds. An important problem in three-manifold topology is to determine when a hyperbolic knot admits a Dehn surgery to a non-hyperbolic three-manifold. My work on this problem is discussed in Sections 5 and 6. On the other hand, many important questions about four-manifolds can be formulated in terms of questions about Dehn surgery on links. This approach is central in the approach to the Poincaré Conjecture discussed in Section 1. In Section 7, I discuss how questions about Dehn surgery on links interact with questions about trisections of homotopy four-spheres and the famous Generalized Property R Conjecture.



1. Toward the Poincaré Conjecture

The Smooth, Four-Dimensional Poincaré Conjecture asserts that every four-manifold with the homotopy-type of the four-sphere is diffeomorphic to the four-sphere. This is the last remnant of (a modernized version of) a 100-year-old conjecture of Herny Poincaré, and is arguably the most important open problem in low-dimensional topology. There has been virtually no progress towards proving this conjecture, though many potential counter-examples have been introduced and eventually dismissed.[2] (For example, see [1, 21, 40].) In light of this, a reasonable approach to the conjecture is to restrict attention to certain types of handle decompositions, as in the following conjecture.

Geometrically Simply-Connected Poincaré Conjecture. *If X is a smooth homotopy four-sphere that can be built without 1–handles, then X is diffeomorphic to S^4.*

If X can be built without 1–handles, then X is called *geometrically simply-connected* (GSC). If X is GSC and has the homotopy type of S^4, then X can be completely described by an n–component link L in S^3 with a Dehn surgery to $\#^n(S^1 \times S^2)$. It is the link L that we endeavor to study. We denote the homotopy four-sphere corresponding to L by X_L.

A classical result along these lines is the famous proof by Gabai of the Property R Conjecture [15], which shows that the only knot with an integral Dehn surgery to $S^1 \times S^2$ is the unknot. Thus, Conjecture 1 is true in the case that L has one component. Zupan and I have studied the case where L has two components, which gives novel progress towards a positive resolution of this conjecture. Let $Q_{p,q} = T_{p,q}\#T_{p,-q}$ denote the knot obtained as the connected sum of the torus knot $T_{p,q}$ with its mirror, which we call a *generalized square knot.*

Theorem 1.1 (M.-Zupan [53])**.** *Suppose that $L = Q \cup J$ is a 2–component link with a Dehn surgery to $\#^2(S^1 \times S^2)$.*

(1) If Q is fibered, then J can be slid over Q some number of times to give a knot J' that lies on a surface-fiber for Q.
(2) If $Q = Q_{p,q}$ is a generalized square knot, then such curves J' on the surface-fiber can be completely characterized.
(3) If $Q = Q_{p,2}$, then X_L is diffeomorphic to S^4.

Part (1) of this theorem involves a careful analysis of the surface-bundle obtained by zero-framed Dehn surgery on the fibered knot Q, and combines thin position techniques from Heegaard theory (e.g., those of [71]) with a seminal result of Casson-Gordon that characterizes the monodromy of such surface-bundles when Q is homotopy-ribbon [5] (see below).

The fact that the hypotheses of parts (2) and (3) of Theorem 1.1 are different means that our analysis of generalized square knots allows us to produce new, concrete, infinite families of potential counter-examples to the Poincaré Conjecture, the first such examples produced in nearly 30 years. A two-component link representing one such example was given in the introduction.

There is another important conjecture lurking in the background here – the Generalized Property R Conjecture. The links $Q \cup J'$ appearing in Theorem 1.1 represent a robust family of potential counter-examples to this conjecture, which was first studied by Gompf, Scharlemann, and Thompson [22]. In Section 6, I describe a program to disprove the Generalized Property R Conjecture that is based in the theory of trisections, which is discussed in Section 2.

[2] Actually, it is not the case that all of the classical examples have been shown to be diffeomorphic to S^4, but convincingly large portions of them have been.

1.1. Fibered, homotopy-ribbon knots.

Recall that a knot (S^3, K) is *slice* if $(S^3, K) = \partial(B, D)$, where D is a properly-embedded disk in a homotopy four-ball B. A slice knot is *ribbon* if it $B \cong B^4$ and if D can be built with only 0–handles and 1–handles inside B^4. A slice knot is *homotopy-ribbon* if there exists a surjection from $\pi_1(S^3 \setminus K)$ onto $\pi_1(B \setminus D)$. Note that ribbon implies homotopy-ribbon implies slice, but the converse implications are the subject of the famous and still-open Slice-Ribbon Conjecture [10].

Because every component of a link L describing a homotopy four-sphere X_L is homotopy-ribbon, as a corollary to Theorem 1.1, we obtain the following classification of fibered, homotopy-ribbon disk-knots bounded by generalized square knots. In previously work with Kyle Larson, I studied fibered, homotopy-ribbon disk knots and their connection with fibered 2–knots (knotted spheres) in homotopy four-spheres [43].

Corollary 1.2 (M.-Zupan [53])**.** *Every generalized square knot $Q_{p,2}$ bounds a unique fibered, homotopy-ribbon disk-knot up to diffeomorphism and this disk-knot is ribbon. Any two fibered, homotopy-ribbon disk-knots bounded by a generalized square knot $Q_{p,q}$ have diffeomorphic exteriors.*

2. Trisections and their classification

Trisections are an exciting new development in the theory of 4–manifolds. Introduced by Gay and Kirby in 2012 [16], they provide a 4–dimensional analog to the concept of Heegaard splittings of 3–manifolds, and there is evidence that they will serve as an effective platform for translating well-established and effective 3–manifold techniques to the realm of 4–manifolds.

Definition 2.1. Let X be a closed, connected, orientable, smooth 4–manifold. A (g, k)*–trisection* of X is a decomposition $X = X_1 \cup X_2 \cup X_3$ such that

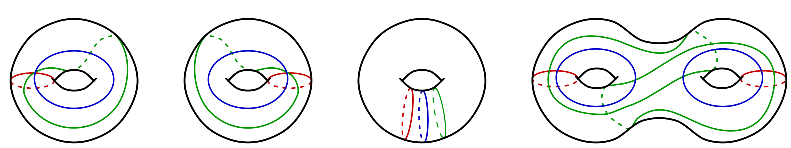
- $X_i \cong \natural^k(S^1 \times B^3)$ is a 4–dimensional handlebody for each i;
- $H_{ij} = X_i \cap X_j \cong \natural^g(S^1 \times D^2)$ is a 3–dimensional handlebody for each pair i, j; and
- $\Sigma = X_1 \cap X_2 \cap X_3$ is a closed surface of genus g.

The surface Σ is called the *trisection surface*, and we say that the trisection has *genus* g.

Notice that $\partial X_i \cong \#^k(S^1 \times S^2)$ and that (Σ, H_{ij}, H_{ki}) is a genus g Heegaard splitting of ∂X_i. In fact, the trisection is completely determined by the quadruple $(\Sigma, H_{12}, H_{23}, H_{31})$ thanks to [44], which says that $\#^k(S^1 \times S^2)$ can be capped off by a 4–dimensional handlebody in a unique way. It's also worth remarking that genus g Heegaard splittings of $\#^k(S^1 \times S^2)$ are known to be standard by [75]. Thus, each pair of 3–dimensional handlebodies determines a trivial Heegaard splitting, but the combination of the three 3–dimensional handlebodies is enough to capture the complexity of the class of smooth 4–manifolds. Note that trisections are a restrictive version of Heegaard triples, which form a cornerstone within the theory of Heegaard Floer homology [61, 65, 64, 66, 67].

Theorem 2.2 (Gay-Kirby [16])**.** *Every closed, connected, orientable, smooth 4–manifold X admits a (g, k)–trisection for some $g \geq k \geq 0$, and any two trisections of X have a common stabilization.*

The only 4–manifold that admits a genus zero trisection is S^4, which can be represented as the union of three 4–balls that intersect pairwise along a 3–ball portion of their S^3 boundaries. It's an easy exercise to determine that there are only three genus one trisections, which correspond to $\mathbb{CP}^2$, $\overline{\mathbb{CP}}^2$, and $S^1 \times S^3$. There is also a nice genus two trisection of $S^2 \times S^2$. These diagrams are shown to the right.



Naturally, one of the first questions to arise following the introduction of trisections was to what extent one could classify trisections of low genus. To this end, Zupan and I proved the following theorem.

Theorem 2.3 (M.-Zupan [55])**.** *Every genus two trisection is equivalent to either the standard trisection of $S^2 \times S^2$ or to a connected sum of the standard genus one trisections.*

The most interesting aspect of this result is that the methods of proof are entirely 3–dimensional. We make use of a result of Homma, Ochiai, and Takahashi that states that every nontrivial genus two Heegaard diagram for S^3 contains a wave [33] and employ a complicated combinatorial argument to show that any nonstandard genus two trisection diagram can be simplified.

2.1. **Additivity of trisection genus.**

We define the *trisection genus* of a 4–manifold X to be

$$g(X) = \min\{g \mid X \text{ admits a } (g,k)\text{–trisection}\}.$$

By the famous Haken's Lemma, the analogous complexity measure for 3–manifolds (Heegaard genus) is additive under connected sums [27]. The most fascinating conjecture about trisections asserts that trisections behave under connected sum in the same manner as Heegaard splittings.

Conjecture 2.4. *Trisection genus is additive under connected sum: If $X = X_1 \# X_2$, then*

$$g(X) = g(X_1) + g(X_2).$$

A positive solution to this conjecture would provide a panacea for 4–manifold topology, as I will now explain. Suppose that X and X' are homeomorphic 4–manifolds. By Wall [76] and Gompf [20], we know that $X \# S$ and $X' \# S$ are diffeomorphic, where $S = \#^n(S^2 \times S^2)$. If trisection genus is additive, then this implies that $g(X) = g(X')$. Since S^4, $\pm\mathbb{CP}^2$, $S^1 \times S^3$, $S^2 \times S^2$ are the only manifolds that admit irreducible[3] trisections with genus at most two [55], this implies that none of these manifolds admits an exotic smooth structure; in particular, the Smooth 4–Dimensional Poincaré Conjecture would hold.

Of course, a proof of Conjecture 2.4 is completely out of reach for the time being; however, it does have the favorable property of casting a host of classical problems in a new light that would not have been apparent before the advent of trisections. For now, the best way to approach this conjecture is to test it to the extent of our abilities. For example, Peter Lambert-Cole and I have shown that many simply connected complex surfaces admit $(g,0)$–trisections [42]. (See the next section for more discussion.) Many of these surfaces are homeomorphic, but not diffeomorphic to standard 4–manifolds, which themselves admit $(g,0)$–trisections. Thus, Conjecture 2.4 is not contradicted in these cases.

An important testing ground for Conjecture 2.4 is the class of 4–manifolds with trisection genus three. It is known that $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}}^2$ admits exotic copies [2], so if the conjecture is to hold, these manifolds will admit $(3,0)$–trisections. In [51], I produced a conjectural list of $(3,1)$–trisections and gave examples of infinitely many manifolds admitting minimal (g,k)–trisections for each $3 \leq g$ and $1 \leq k \leq g-1$.

To this end, Alexandra Kjuchukova and I have a program to generate 4–manifolds admitting $(3,0)$–trisections by taking dihedral branched covers of singular surfaces in S^4 and in $\mathbb{CP}^2$. Our techniques could prove useful in determining whether or not there are small exotic manifolds admitting such trisections, as discussed in the previous section.

[3]A trisection is *reducible* if it is the connected sum of two trisections

3. Bridge trisections of knotted surfaces

Given a 3–dimensional handlebody $H \cong \natural^g(S^1 \times D^2)$, a *trivial b–tangle* $\tau \subset H$ is a collection of b properly embedded, disjoint arcs that can be simultaneously isotoped to lie in ∂H. Given a 4–dimensional 1–handlebody $Z \cong \natural^k(S^1 \times D^2)$, a *trivial c–disk-tangle* $\mathcal{D} \subset Z$ is a collection of c properly embedded, disjoint disks that can be simultaneously isotoped rel–∂ to lie in ∂Z.

Definition 3.1. A closed surface $\mathcal{S}$ in a closed 4–manifold X is in (b,c)*–bridge trisected position* with respect to a trisection $\mathfrak{T}$ of X if

- $\mathcal{D}_i = X_i \cap \mathcal{S}$ is a trivial c–disk-tangle for each i; and
- $\tau_{ij} = H_{ij} \cap \mathcal{S}$ is a trivial b–tangle for each pair i,j.

The surface $\Sigma = X_1 \cap X_2 \cap X_3$ is called the *bridge surface*. The decomposition

$$(X, \mathcal{S}) = (X_1, \mathcal{D}_1) \cup (X_2, \mathcal{D}_2) \cup (X_3, \mathcal{D}_3)$$

is called a *bridge trisection*.

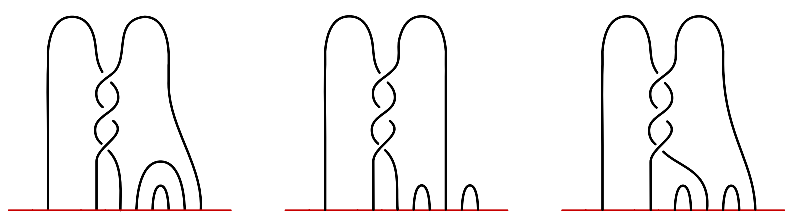
Any two trivial c–disk-tangles in Z whose boundary agree are isotopic rel–∂ [35, 46, 56], so it follows that a bridge trisection is completely determined by the triple of trivial tangles (H_{ij}, α_{ij}).

Theorem 3.2 (M.-Zupan [54, 56])**.** *Let $\mathcal{S}$ be a closed surface in a 4–manifold X, and let $\mathfrak{T}$ be a given trisection of X. Then $\mathcal{S}$ can be isotoped to lie in (b,c)–bridge trisected position for some $b \geq c \geq 1$. Let $\mathfrak{T}_0$ be the genus zero trisection of S^4. Any two bridge trisections of a surface $\mathcal{S}$ in S^4 with respect to $\mathfrak{T}_0$ are stably equivalent.*

Note that $\mathcal{S}$ is not required to be connected nor orientable. Three particular settings warrant special attention: (a) The case of the genus zero trisection $\mathfrak{T}_0$ of S^4; (b) The case of knotted spheres (*2–knots*) in arbitrary 4–manifolds; and (c) The case of complex curves in complex surfaces.

3.1. Tri-plane diagrams for surface-links in 4–space.

Fix S^4 as the ambient 4–manifold and fix the genus zero trisection $\mathfrak{T}_0$. In this case the handlebodies are 3–balls, and a bridge trisection can be represented diagrammatically by projecting the strands of the three trivial tangles onto the union of the equatorial disks of the tangles. The resulting collection of tangle diagrams is called a *tri-plane diagram*: a triple of trivial b–strand tangle diagrams whose pairwise unions are unlinks of c–components. For example, a tri-plane diagram for the spun trefoil is shown to the right.



Theorem 3.3 (M.-Zupan [54])**.** *Every closed surface $\mathcal{S}$ in S^4 can be represented by a (b,c)–tri-plane diagram for some $b \geq c \geq 1$, and any two diagrams for $\mathcal{S}$ are related by a finite sequence of tri-plane moves.*

Bridge trisections are closely related to trisections of 4–manifolds. Let $\mathcal{B}$ be a (b,c)–bridge trisection of a surface $\mathcal{S}$ in S^4, and let $X_{\mathcal{S}}$ denote the double cover of S^4, branched along $\mathcal{S}$. Then $X_{\mathcal{S}}$ admits a $(b-1, c-1)$–trisection whose pieces are simply the branched double covers of the pieces of $\mathcal{B}$. By [26], every genus two handlebody is the branched double cover of a trivial 3–strand tangle, so we get the following corollary to Theorem 2.3.

Corollary 3.4. *Every (b,c)–bridge trisection with $b \leq 3$ is standard, and the surface is unknotted.*

In [54], we also construct knotted surfaces with minimal bridge trisections by spinning torus knots, giving infinitely many distinct 2–knots with minimal $(3p-2,p)$–bridge trisections for each $p \geq 2$. Interestingly, all known examples of 2–knots with 4–bridge trisections are unions of ribbon disks for 2–bridge knots, which motivates the following problem.

Problem 3.5. *Classify 4–bridge trisections of 2–knots.*

Another interesting class of surfaces are those that are orientable and admit $(b,1)$–bridge trisections. By topological surgery theory [12, 13, 30, 36, 38, 39], such surfaces are topologically unknotted, since their complements have infinite cyclic fundamental group. The Unknotting Conjecture [37] states that any such surface is smoothly unknotted.

Question 3.6. *Is there a smoothly knotted surface whose complement has infinite cyclic fundamental group? Can such a surface admit a $(b,1)$–bridge trisection for some b?*

3.2. Gluck twist and surgery on 2–knots.

One of the most important reasons for studying knotted surfaces in four-space is to understand the four-manifolds obtained from various surgery operations on these knotted surfaces. For example, the *Gluck twist* is the operation given by removing a neighborhood of a 2–knot and re-gluing using the unique non-trivial diffeomorphism of the boundary. The result of a Gluck twist on a 2–knot is easily seen to be a homotopy four-sphere, but it is unknown if such manifolds are diffeomorphic to S^4 [19]. When $\mathcal{S}$ is a 2–knot, we have the following.

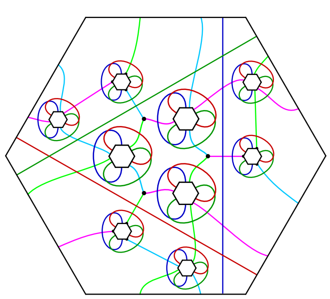
Theorem 3.7 (M.-Zupan [56])**.** *Every 2–knot $(X,\mathcal{K})$ can be represented by a doubly-pointed trisection diagram.*

In joint work with David Gay, we gave an analysis of various surgery operations on 2–knots. We also proved a uniqueness statement that corresponds with the existence theorem above.

Theorem 3.8 (Gay-M. [17])**.** *Any two doubly-pointed trisection diagrams for a given 2–knot $(X,\mathcal{K})$ are stably equivalent. There is a straight-forward procedure to produce a trisection diagram for a 4–manifold $X(\mathcal{K})$ obtained by performing a Gluck twist, surgery, or a rational blow-down along $\mathcal{K}$ in X.*

3.3. Complex curves in complex surfaces.

The theories of trisections and bridge trisections are elegantly related by branched covering constructions. This is because the branched cover of a bridge trisected knotted surface naturally gives rise to a trisection on the cover. Lambert-Cole and I have been using this principal to study trisections of complex surfaces and bridge trisections of complex curves therein. We have conjectured that such objects admit the simplest possible sort of trisections. A (g,k)–trisection of a 4–manifold X is called *efficient* if $k = \text{rank}(\pi_1(X))$. Similarly, a bridge trisection for $(X,\mathcal{K})$ is called *efficient* if the underlying trisection of X is efficient and if $c = \text{mer}(\pi_1(X \setminus \mathcal{K}))$ (the meridional rank). To the left, we see an efficient bridge trisection of the torus fiber inside the elliptic surface $E(1)$.



It is an open problem to determine if every 4–manifold admits an efficient trisection; this is related to the question of whether every simply connected 4–manifold can be built without 1–handles. While it may be too much to ask for things to work out so nicely in general, there is evidence that restricting the geometry of the 4–manifold can help.

Conjecture 3.9. *If X is a simply connected complex surface and $\mathcal{C} \subset X$ is a complex curve with $\pi_1(X \setminus \mathcal{C})$ cyclic, then the pair $(X,\mathcal{C})$ admits an efficient bridge trisection.*

Lambert-Cole and I have begun to verify this conjecture in a handful of cases.

Theorem 3.10 (Lambert-Cole-M. [42])**.** *Every complex curve in $\mathbb{CP}^2$, $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, or $\mathbb{CP}^1\times\mathbb{CP}^1$ admits an efficient bridge trisection.*

Since any branched covering of a knotted surface admitting an efficient bridge trisection will admit an efficient trisection, this gives an expansive list of complex surfaces that admit efficient trisections. This list includes complete intersections of complex hyper-surfaces in $\mathbb{CP}^N$, elliptic surfaces, and many Horikawa surfaces. Many of these complex surfaces are exotic copies of standard 4–manifolds. Since they admit efficient trisections this offers evidence towards Conjecture 2.4 regarding the additivity of trisection genus. A trisection of the K3 surface is shown below, after the references.

4. Doubly slice knots

In 1962, Fox included the following question in his list of problems in knot theory [10].

Question 4.1. *Which knots can appear as the cross-section of the unknotted S^2 in S^4?*

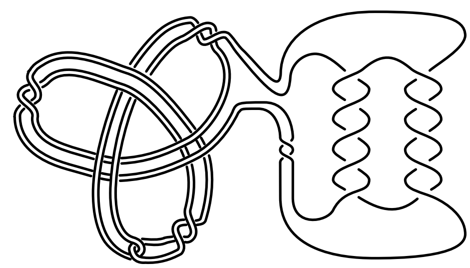
Since then, such knots have come to be called *doubly slice*. If one allows the S^2 to be knotted in S^4, then the cross-section is simply called *slice*. Over the last 50 years, many advances in low-dimensional topology have led to breakthroughs in the study of slice knots, while the study of doubly slice knots has remained must less well-explored. In fact, there is the following technical obstruction to studying doubly slice knots in direct analogy with slice knots.

Question 4.2. *Do there exist knots K and J such that K and $K\#J$ are doubly slice, but J is not?*

With Question 4.2 in mind, we say that K_0 and K_1 are *smoothly doubly concordant* if there exist smoothly doubly slice knots J_0 and J_1 such that $K_0\#J_0$ and $K_1\#J_1$ are ambient isotopic. Under connected sum, the collection of knots in S^3 modulo smooth double concordance forms an abelian group, denoted $\mathcal{C}_\mathcal{D}$, called the *smooth double concordance group*. Analogously, we can form the *topological* and *algebraic double concordance groups*, $\mathcal{C}_\mathcal{D}^{top}$ and $\mathcal{G}_\mathcal{D}$. These groups can be organized nicely via natural surjections:

$$\mathcal{C}_\mathcal{D} \xrightarrow{\Psi} \mathcal{C}_\mathcal{D}^{top} \xrightarrow{\Phi} \mathcal{G}_\mathcal{D}.$$

In 1983, Gilmer and Livingston gave the first examples of slice knots that are algebraically doubly slice, but not doubly slice [18], and more recently, Kim [41] extended the techniques of Cochran, Orr, and Teichner to show that the collection of such knots has a rich filtered structure. The seminal work of Freedman [12, 11] and Donaldson [8] in the early 1980s, illuminated the difference between the smooth and topological categories in dimension four. Immediately, these differences were detected in the study of slice knots [7], and the blossoming of knot Floer homology [62, 63, 68, 69] has recently led to an improved understanding of this difference (e.g., [28, 29, 32, 60]).



For many years, nothing was known about the distinction between the smooth and topologically locally flat categories for doubly slice knots. However, in [49], I was able to prove theorems illustrating this distinction.

Theorem 4.3 (M. [49])**.** *There exists an infinite family of smoothly slice knots that are topologically doubly slice but not smoothly doubly slice.*

These knots are formed by infecting smoothly doubly slice knots with Whitehead doubles of the trefoil, as in the picture above. A geometric argument shows the resulting knots are still topologically doubly slice (and smoothly slice), and a calculation of the Heegaard Floer homology correction terms [61] using the surgery exact sequence [68, 69] shows that the double branched covers cannot embed smoothly in S^4. Thus, the knots cannot be smoothly doubly slice.

The proof has the following interesting corollary.

Corollary 4.4. *There exists an infinite family of rational homology 3–spheres with nontrivial first homology that embed in S^4 topologically but not smoothly.*

All previously known examples of such objects were integer homology spheres [12, 70] and didn't bound smooth rational homology four-balls; the present examples have nontrivial first homology and bound smooth rational homology balls. This points to a 3–manifold version of Question 4.2.

Question 4.5. *Do there exist closed 3–manifolds M and N such that M and $M\#N$ embed smoothly in S^4, but N does not?*

Though Question 4.2 remains open, my work shows that the correction terms actually provide obstructions to a knot being *stably* doubly slice, which allows for a stronger restatement of Theorem 4.3.

Theorem 4.6 (M. [49])**.** *There is an infinitely generated subgroup $\mathcal{S}$ inside $\ker(\Psi)$ that is generated by smoothly slice knots whose order in $\mathcal{C}_{\mathcal{D}}$ is at least three.*

It seems reasonable to conjecture that $\mathcal{S}$ describes a free abelian direct summand of $\mathcal{C}_{\mathcal{D}}$. Nothing else is known about the structure of the kernels of Ψ and Φ, and a primary goal of my future research is to explore these objects further. Along these lines, we can ask the following question.

Question 4.7. *What is the structure of $\ker(\Phi)$ and $\ker(\Psi)$?*

Other than what has been described above, very little is known about these subgroups. It would be nice to find elements of finite order in either kernel. Additionally, a conjecture of Gordon (see [24]) states that every element of order 2 in $\mathcal{C}$ is represented by a negative-amphicheiral knot. It is interesting to wonder if a similar statement holds for elements of $\mathcal{C}_{\mathcal{D}}$.

A natural first step to trying to answer these important questions is to gain a firm understanding of basic examples of doubly slice knots. To this end, Chuck Livingston and I gave an analysis of doubly slice knots up to twelve crossings [47], building on Sumners' earlier work [72].

Theorem 4.8 (Livingston-M. [47])**.** *Among all knots with at most 12 crossings, all but three knots can be shown to be either smoothly doubly slice or not topologically doubly slice, and only the Conway knot, 10_{n34}, is known to be topologically doubly slice but not smoothly doubly slice.*

Theorem 4.8 leaves open four cases for future study, and gives motivation to expand the classification beyond twelve crossings. The techniques involved range from classical algebraic techniques to modern topological techniques involving twisted Alexander polynomials. Interestingly, smooth invariants failed to provide assistance for these low-crossing knots but will likely be useful in the future.

5. Cabled slice knots and reducible surgeries

Dehn surgery is one of the simplest and most important operations in 3–manifold topology. In short, Dehn surgery is the process of removing a solid torus neighborhood of a knot and gluing it back in with a different identification, called a slope. One of the most basic open questions asks when the result of Dehn surgery can be a reducible manifold (a manifold containing an essential two-sphere). Let $J_{p,q}$ denote the (p,q)–cable of J, and let $S^3_r(K)$ denote the result of performing Dehn surgery on K along slope $r \in \mathbb{Q}$. The Cabling Conjecture [23] asserts that every reducible Dehn surgery is pq–surgery on a (p,q)–cabled knot. A more accessible version of this very hard conjecture is the following.

Two Summands Conjecture. *If K is a nontrivial knot in S^3 and $S^3_r(K)$ is reducible, then $S^3_r(K) \cong Y_1\#Y_2$, with Y_1 and Y_2 irreducible.*

The Two-Summands Conjecture is true for knots with bridge number at most five and positive braid closures [82]. In [50], I verified the Two Summands Conjecture for slice knots.

Theorem 5.1 (M. [50]). *A slice knot in the 3–sphere cannot admit a reducible surgery with three irreducible summands.*

In fact, Theorem 5.1 holds for any knot K with $V_0(K) = V_0(\overline{K}) = 0$, where $V_0(K)$ is a Heegaard Floer theoretic knot invariant coming from the knot Floer complex that determines the correction terms of surgeries on K [68, 69], and $\overline{K}$ denotes the mirror of K. The condition that $V_0(K) = 0$ suffices in the case that the surgery is positive. Along these same lines, we have the following observation.

Theorem 5.2 (M. [50]). *Let $K = J_{p,q}$ with $p, q > 0$. If K is slice, then $V_0(J) = 0$.*

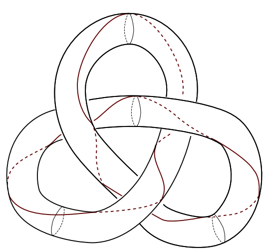
A key insight in the proof is the following fact, which can be proved using classical methods: If $J_{p,q}$ is algebraically slice, then $q = 1$. This leads to the following natural conjecture.

Conjecture 5.3. *The cabled knot $J_{p,q}$ is slice if and only if J is slice and $q = 1$.*

Since any $(p, 1)$–cable of slice knot is also slice, one direction of the conjecture is true. Theorem 5.2, along with [31] and [81], give evidence that the conjecture is true at the level of Heegaard Floer homology. Homology invariants of cabled knots have also been studied recently in [6, 9]. Note that Conjecture 5.3 is true for fibered knots in the homotopy-ribbon setting by Theorem 8.5 of [58].

6. Exceptional Seifert fibered surgery

Around 1960, Lickorish and Wallace proved that every 3–manifold can be obtained by Dehn surgery on some link in S^3 [45, 77]. Since then, a major problem in low-dimensional topology has been to understand the special case of Dehn surgery on knots in S^3. By Thurston's Geometrization for knot complements [74], we know that every knot is either a torus knot, a satellite knot, or a hyperbolic knot. Surgeries on torus knots are well-understood [59], as are surgeries on satellite knots [25]. Moreover, Thurston showed that a hyperbolic knot can only have finitely many surgeries that produce non-hyperbolic 3–manifolds [73]. Such surgeries are called *exceptional*, and have been the subject of decades of study.



Exceptional surgeries can be divided into three categories: reducible, toroidal, and Seifert fibered. While open questions remain in all cases, understanding the case of Seifert fibered surgeries has presented a particularly formidable challenge over the years, and very little is known or even conjectured about such surgeries. Seifert fibered spaces split naturally into three cases: lens spaces, toroidal Seifert fibered spaces, and small Seifert fibered spaces.

It is natural to ask which hyperbolic knots admit Seifert fibered surgeries. Building on work of Wu and Brittenham-Wu [4, 78, 79, 80], I was able to give a classification of small Seifert fibered surgeries on hyperbolic pretzel knots, as well as a near-classification for Montesinos knots. Together with [34], this gives a complete classification for a large class of knots called arborescent knots.

Theorem 6.1 (M. [48], Ichihara-Masai [34], Brittenham-Wu [4], Wu [78]). *There is a complete classification of small Seifert fibered surgeries on hyperbolic arborescent knots.*

Question 6.2. *Can a toroidal Seifert fibered space be realized as Dehn surgery on a hyperbolic knot?*

One of the most intriguing (and difficult) conjectures about exceptional Dehn surgery asserts that every Seifert fibered surgery is integral. The work described above verifies this conjecture for arborescent knots.

Corollary 6.3. *Every Seifert fibered surgery on a hyperbolic arborescent knot is integral.*

7. Trisections and the Generalized Property R Conjecture

There is a slight generalization of trisections called *unbalanced* trisections. Here, we allow the complexity of the three pieces X_i to vary, requiring only that $X_i \cong \natural^{k_i}(S^1 \times B^3)$ for some $k_i \geq 0$. Such a decomposition is called a $(g; k_1, k_2, k_3)$*–trisection.* For example, there are three unbalanced genus one trisections.

Theorem 7.1 (M.-Schirmer-Zupan [52])**.** *Every $(g; k_1, k_2, k_3)$–trisection with $k_1 \geq g-1$ is equivalent to a connected sum of (balanced or unbalanced) genus one trisections.*

The most interesting aspect of the theorem is the way in which 3–manifold techniques are used to prove a 4–dimensional result. In this case, we make use of the 3–fold symmetry inherent to trisections, called *handle triality.* This idea is analogous to the classical approach of analyzing the dual description of a manifold by turning it upside down. Handle triality puts 1–, 2–, and 3–handles on equal footing and allows for a 4–manifold with a trisection to be analyzed in *six* different ways.

For example, a $(g; 0, g-1, 1)$–trisection gives rise to a Dehn surgery on knot in S^3 that yields $S^1 \times S^2$. Such a surgery is trivial, by Gabai [14]. On the other hand, by permuting the parameters, we get a $(g; 0, 1, g-1)$–trisection, which gives rise to a c–component link in S^3 with a surgery to $\#^c(S^1 \times S^2)$, an object that first showed up in Section 1. Such links are the subject of a major conjecture; we now state a stable version of this conjecture that is most appropriate in this setting.

(Stable) Generalized Property R Conjecture. *Suppose that L is a c–component link in S^3 with a surgery to $\#^c(S^1 \times S^2)$. Let L' be the split union of L with an r–component unlink for some $r \geq 0$. Then there exists a sequence of handleslides converting L' into a $(c+r)$–component unlink.*

Thus, by permuting the order of the parameters, we can re-phrase trisection problems in terms of different types of 3–manifold questions, some that have been solved and some that have not. A good example of this is the following corollary.

Corollary 7.2 (M.-Schirmer-Zupan [52])**.** *Let L be a c–component link in S^3 with an integral Dehn surgery to $\#^c(S^1 \times S^2)$. If L has tunnel number at most c, then L satisfies the (Stable) Generalized Property R Conjecture.*

Gompf, Scharlemann, and Thompson [22] produced the first potential counterexamples to this conjecture and showed that if their links satisfy the conjecture then the presentations P_n of the trivial group given by

$$\langle x, y \mid xyx = yxy, x^n = y^{n+1} \rangle$$

are Andrews-Curtis trivial [3], which is widely believed to be false for $n \geq 3$. Zupan and I gave significant generalizations of this work in [53] and made explicit a deep connection between the (Stable) Generalized Property R Conjecture and trisections, building on a program initiated in [53]. Given an R-link L, there is a natural way to obtain a trisection $\mathfrak{T}_L$ for the homotopy four-sphere X_L. (See Sectoin 1.

Theorem 7.3 (M.-Zupan [53, 57])**.** *The link L satisfies the (Stable) Generalized Property R Conjecture if and only if the trisection $\mathfrak{T}_L$ becomes completely destabilizable after some number of stabilizations that only increase the parameter k_2.*

8. Annotated Bibliography

In *Genus two trisections are standard* [55], Alexander Zupan and I classify which 4–manifolds admit genus two trisections. The proof is entirely 3–dimensional, giving evidence that the theory of trisections will be able to successfully promote 3–dimensional techniques to solve 4–dimensional problems.

In *Classification of trisections and the Generalized Property R Conjecture* [52], Trent Schirmer, Zupan, and I give classification results for a family of unbalanced trisections. One corollary to this work is a relationship between the Generalized Property R Conjecture (and, hence, the Andrews-Curtis Conjecture) and the problem of simplifying trisections of S^4.

In *Bridge trisections of knotted surfaces in* S^4 [54], Zupan and I adapt the theory of trisections to the setting of knotted surfaces in S^4. The resulting decompositions, called bridge trisections, are analogous to bridge splittings of knots and links in S^3, and offer a new measure of complexity for knotted surfaces, as well as a new diagrammatic theory for knotted surfaces.

In *Bridge trisections of knotted surfaces in four-manifolds* [56], Zupan and I extend the theory of bridge trisections to arbitrary four-manifolds, giving a diagrammatic approach to this broad area of four-dimensional topology. We find that complex curves in $\mathbb{CP}^2$ admit efficient bridge trisections.

In *Characterizing Dehn surgery on links via trisections* [57], Zupan and I formalize the connection between Dehn surgery and trisections and give a program to disprove the Generalized Property R Conj.

In *Trisections and spun 4–manifolds* [51], I give a rich list of examples of minimal trisections, pushing the connection between three-dimensional and four-dimensional techniques.

In *Distinguishing topologically and smoothly doubly slice knots* [49], I use the Heegaard Floer homology correction terms to exhibit an infinite family of smoothly slice knots that are topologically doubly slice but not smoothly doubly slice. One corollary is that there exists an infinite family of rational homology spheres that embed in S^4 topologically but not smoothly.

In *Doubly slice knots with low crossing number* [47], Chuck Livingston and I attempt to catalogue all doubly slice knots up to 12 crossings. We were able to exhibit 18 new smoothly doubly slice knots, and obstruct all but four of the remaining knots from being topologically doubly slice.

In *A note on cabled slice knots and reducible surgeries* [50], I show that a (p, q)–cable of a knot is algebraically slice, then $q = 1$ and that if it is slice, then a certain Heegaard Floer invariant must vanish for the companion knot. I also show that Dehn surgery on a slice knot cannot produce a 3–manifold with three irreducible summands.

In *Fibered ribbon disks* [43], Kyle Larson and I give a characterization of fibered homotopy-ribbon disks as those whose fiber is a handlebody and introduce an analog of the Stallings twist for such fibrations, which we interpreted in terms of 4–dimensional surgery operations. Our twisting operation can be used to produce potential counterexamples to the Slice-Ribbon Conjecture.

In *Small Seifert fibered surgery on hyperbolic pretzel knots* [48], I classify Seifert fibered surgeries on hyperbolic Montesinos knots. As a corollary, I verify that Montesinos knots satisfy the Seifert Fibered Space Conjecture, which posits that all exceptional Seifert fibered space surgeries are integral.

In *Doubly pointed trisection diagrams and surgery on 2–knots* [17], David Gay and I give a method for trisecting the complement of a 2–knot (in any four-manifold) and show how one can obtain trisection diagrams for the four-manifolds obtained via the Gluck twist and surgery operations.

In *Generalized square knots and homotopy 4–spheres* [53], Zupan and I show that a homotopy 4–sphere that is built with two 2–handles and no 1–handles is necessarily standard if one of the 2–handles is attached along the generalized square knot $Q_{p,2} = T_{p,2} \# T_{p,-2}$.

In *Bridge trisections in rational surfaces* [42], Lambert-Cole and I show that every complex curve in $\mathbb{CP}^2$, $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, or $\mathbb{CP}^1 \times \mathbb{CP}^1$ admits an efficient bridge trisection. As a corollary, we give many examples of pairs of exotic manifolds that admit efficient trisections hence have the same trisection genus.

REFERENCES

[1] AKBULUT, S. Cappell-Shaneson homotopy spheres are standard. *Ann. of Math. (2) 171*, 3 (2010), 2171–2175.

[2] AKHMEDOV, A., AND PARK, B. D. Exotic smooth structures on small 4-manifolds with odd signatures. *Invent. Math. 181*, 3 (2010), 577–603.

[3] ANDREWS, J. J., AND CURTIS, M. L. Free groups and handlebodies. *Proc. Amer. Math. Soc. 16* (1965), 192–195.

[4] BRITTENHAM, M., AND WU, Y.-Q. The classification of exceptional Dehn surgeries on 2-bridge knots. *Comm. Anal. Geom. 9*, 1 (2001), 97–113.

[5] CASSON, A. J., AND GORDON, C. M. A loop theorem for duality spaces and fibred ribbon knots. *Invent. Math. 74*, 1 (1983), 119–137.

[6] CHEN, W. On the Upsilon invariant of cable knots. arXiv:1604.04760, 2016.

[7] COCHRAN, T. D., AND GOMPF, R. E. Applications of Donaldson's theorems to classical knot concordance, homology 3-spheres and property P. *Topology 27*, 4 (1988), 495–512.

[8] DONALDSON, S. K. An application of gauge theory to four-dimensional topology. *J. Differential Geom. 18*, 2 (1983), 279–315.

[9] FELLER, P., PARK, J., AND RAY, A. On the Upsilon invariant and satellite knots. arXiv:1604.04901, 2016.

[10] FOX, R. H. Some problems in knot theory. In *Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961)*. Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 168–176.

[11] FREEDMAN, M., AND QUINN, F. A quick proof of the 4-dimensional stable surgery theorem. *Comment. Math. Helv. 55*, 4 (1980), 668–671.

[12] FREEDMAN, M. H. The topology of four-dimensional manifolds. *J. Differential Geom. 17*, 3 (1982), 357–453.

[13] FREEDMAN, M. H., AND QUINN, F. *Topology of 4-manifolds*, vol. 39 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1990.

[14] GABAI, D. Foliations and the topology of 3-manifolds. II. *J. Differential Geom. 26*, 3 (1987), 461–478.

[15] GABAI, D. Foliations and the topology of 3-manifolds. III. *J. Differential Geom. 26*, 3 (1987), 479–536.

[16] GAY, D., AND KIRBY, R. Trisecting 4-manifolds. *Geom. Topol. 20*, 6 (2016), 3097–3132.

[17] GAY, D., AND MEIER, J. Doubly pointed trisection diagrams and surgery on 2-knots. arXiv:1806.05351.

[18] GILMER, P. M., AND LIVINGSTON, C. On embedding 3-manifolds in 4-space. *Topology 22*, 3 (1983), 241–252.

[19] GLUCK, H. The embedding of two-spheres in the four-sphere. *Trans. Amer. Math. Soc. 104* (1962), 308–333.

[20] GOMPF, R. E. Stable diffeomorphism of compact 4-manifolds. *Topology Appl. 18*, 2-3 (1984), 115–120.

[21] GOMPF, R. E. Killing the Akbulut-Kirby 4-sphere, with relevance to the Andrews-Curtis and Schoenflies problems. *Topology 30*, 1 (1991), 97–115.

[22] GOMPF, R. E., SCHARLEMANN, M., AND THOMPSON, A. Fibered knots and potential counterexamples to the property 2R and slice-ribbon conjectures. *Geom. Topol. 14*, 4 (2010), 2305–2347.

[23] GONZÁLEZ-ACUÑA, F., AND SHORT, H. Cyclic branched coverings of knots and homology spheres. *Rev. Mat. Univ. Complut. Madrid 4*, 1 (1991), 97–120.

[24] GORDON, C. M. Some aspects of classical knot theory. In *Knot theory (Proc. Sem., Plans-sur-Bex, 1977)*, vol. 685 of *Lecture Notes in Math.* Springer, Berlin, 1978, pp. 1–60.

[25] GORDON, C. M. Dehn surgery and satellite knots. *Trans. Amer. Math. Soc. 275*, 2 (1983), 687–708.

[26] HAAS, A., AND SUSSKIND, P. The geometry of the hyperelliptic involution in genus two. *Proc. Amer. Math. Soc. 105*, 1 (1989), 159–165.

[27] HAKEN, W. Some results on surfaces in 3-manifolds. In *Studies in Modern Topology*. Math. Assoc. Amer. (distributed by Prentice-Hall, Englewood Cliffs, N.J.), 1968, pp. 39–98.

[28] HEDDEN, M., KIM, S.-G., AND LIVINGSTON, C. Topologically slice knots of smooth concordance order two. *J. Differential Geom. 102*, 3 (2016), 353–393.

[29] HEDDEN, M., LIVINGSTON, C., AND RUBERMAN, D. Topologically slice knots with nontrivial Alexander polynomial. *Adv. Math. 231*, 2 (2012), 913–939.

[30] HILLMAN, J. A., AND KAWAUCHI, A. Unknotting orientable surfaces in the 4-sphere. *J. Knot Theory Ramifications 4*, 2 (1995), 213–224.

[31] HOM, J. Bordered Heegaard Floer homology and the tau-invariant of cable knots. *J. Topol. 7*, 2 (2014), 287–326.

[32] HOM, J. An infinite-rank summand of topologically slice knots. *Geom. Topol. 19*, 2 (2015), 1063–1110.

[33] HOMMA, T., OCHIAI, M., AND TAKAHASHI, M.-O. An algorithm for recognizing S^3 in 3-manifolds with Heegaard splittings of genus two. *Osaka J. Math. 17*, 3 (1980), 625–648.

[34] ICHIHARA, K., AND MASAI, H. Exceptional surgeries on alternating knots. arXiv:1310.3472, 2013.

[35] KAMADA, S. *Braid and knot theory in dimension four*, vol. 95 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[36] KAWAUCHI, A. Splitting a 4-manifold with infinite cyclic fundamental group. *Osaka J. Math. 31*, 3 (1994), 489–495.

[37] KAWAUCHI, A. *A survey of knot theory.* Birkhäuser Verlag, Basel, 1996. Translated and revised from the 1990 Japanese original by the author.

[38] KAWAUCHI, A. Splitting a 4-manifold with infinite cyclic fundamental group, revised. *J. Knot Theory Ramifications 22*, 14 (2013), 1350081, 9.

[39] KAWAUCHI, A. Splitting a 4-manifold with infinite cyclic fundamental group, revised in a definite case. *J. Knot Theory Ramifications 23*, 5 (2014), 1450029, 6.

[40] KIM, M. H., AND YAMADA, S. Ideal classes and Cappell-Shaneson homotopy 4-spheres. *ArXiv e-prints* (July 2017).

[41] KIM, T. New obstructions to doubly slicing knots. *Topology 45*, 3 (2006), 543–566.

[42] LAMBERT-COLE, P., AND MEIER, J. Bridge trisections in rational surfaces. arXiv:1810.10450, 2018.

[43] LARSON, K., AND MEIER, J. Fibered ribbon disks. *J. Knot Theory Ramifications 24*, 14 (2015), 1550066, 22.

[44] LAUDENBACH, F., AND POÉNARU, V. A note on 4-dimensional handlebodies. *Bull. Soc. Math. France 100* (1972), 337–344.

[45] LICKORISH, W. B. R. A representation of orientable combinatorial 3-manifolds. *Ann. of Math. (2) 76* (1962), 531–540.

[46] LIVINGSTON, C. Surfaces bounding the unlink. *Michigan Math. J. 29*, 3 (1982), 289–298.

[47] LIVINGSTON, C., AND MEIER, J. Doubly slice knots with low crossing number. *New York J. Math. 21* (2015), 1007–1026.

[48] MEIER, J. Small Seifert fibered surgery on hyperbolic pretzel knots. *Algebr. Geom. Topol. 14*, 1 (2014), 439–487.

[49] MEIER, J. Distinguishing topologically and smoothly doubly slice knots. *J. Topol. 8*, 2 (2015), 315–351.

[50] MEIER, J. A note on cabled slice knots and reducible surgeries. arXiv:1507.04562; To appear in *Mich. Math. Journal*, 2015.

[51] MEIER, J. Trisections and spun 4–manifolds. arXiv:1708.01214, 2017.

[52] MEIER, J., SCHIRMER, T., AND ZUPAN, A. Classification of trisections and the Generalized Property R Conjecture. *Proc. Amer. Math. Soc. 144*, 11 (2016), 4983–4997.

[53] MEIER, J., AND ZUPAN, A. Generalized square knots and homotopy 4–spheres. *In preparation.*

[54] MEIER, J., AND ZUPAN, A. Bridge trisections of knotted surfaces in S^4. *Trans. Amer. Math. Soc. 369*, 10 (2017), 7343–7386.

[55] MEIER, J., AND ZUPAN, A. Genus-two trisections are standard. *Geom. Topol. 21*, 3 (2017), 1583–1630.

[56] MEIER, J., AND ZUPAN, A. Bridge trisections of knotted surfaces in 4–manifolds. arXiv:1710.01745, To appear in *Proc. Nat. Acad. Sci. U.S.A.*, 2018.

[57] MEIER, J., AND ZUPAN, A. Characterizing Dehn surgeries on links via trisections. arXiv:1707.08955, To appear in *Proc. Nat. Acad. Sci. U.S.A.*, 2018.

[58] MIYAZAKI, K. Nonsimple, ribbon fibered knots. *Trans. Amer. Math. Soc. 341*, 1 (1994), 1–44.

[59] MOSER, L. Elementary surgery along a torus knot. *Pacific J. Math. 38* (1971), 737–745.

[60] OZSVATH, P., STIPSICZ, A., AND SZABO, Z. Concordance homomorphisms from knot floer homology. arXiv:1407.1795, 2017.

[61] OZSVÁTH, P., AND SZABÓ, Z. Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary. *Adv. Math. 173*, 2 (2003), 179–261.

[62] OZSVÁTH, P., AND SZABÓ, Z. Knot Floer homology and the four-ball genus. *Geom. Topol. 7* (2003), 615–639.

[63] OZSVÁTH, P., AND SZABÓ, Z. Holomorphic disks and knot invariants. *Adv. Math. 186*, 1 (2004), 58–116.

[64] OZSVÁTH, P., AND SZABÓ, Z. Holomorphic disks and three-manifold invariants: properties and applications. *Ann. of Math. (2) 159*, 3 (2004), 1159–1245.

[65] OZSVÁTH, P., AND SZABÓ, Z. Holomorphic disks and topological invariants for closed three-manifolds. *Ann. of Math. (2) 159*, 3 (2004), 1027–1158.

[66] OZSVÁTH, P., AND SZABÓ, Z. Holomorphic triangle invariants and the topology of symplectic four-manifolds. *Duke Math. J. 121*, 1 (2004), 1–34.

[67] OZSVÁTH, P., AND SZABÓ, Z. Holomorphic triangles and invariants for smooth four-manifolds. *Adv. Math. 202*, 2 (2006), 326–400.

[68] OZSVÁTH, P. S., AND SZABÓ, Z. Knot Floer homology and integer surgeries. *Algebr. Geom. Topol. 8*, 1 (2008), 101–153.

[69] OZSVÁTH, P. S., AND SZABÓ, Z. Knot Floer homology and rational surgeries. *Algebr. Geom. Topol. 11*, 1 (2011), 1–68.

[70] ROHLIN, V. A. New results in the theory of four-dimensional manifolds. *Doklady Akad. Nauk SSSR (N.S.) 84* (1952), 221–224.

[71] SCHARLEMANN, M., AND THOMPSON, A. A. Surgery on a knot in (surface $\times$ I). *Algebr. Geom. Topol. 9*, 3 (2009), 1825–1835.

[72] SUMNERS, D. W. Invertible knot cobordisms. *Comment. Math. Helv. 46* (1971), 240–256.

[73] THURSTON, W. Hyperbolic geometry and 3-manifolds. In *Low-dimensional topology (Bangor, 1979)*, vol. 48 of *London Math. Soc. Lecture Note Ser.* Cambridge Univ. Press, Cambridge-New York, 1982, pp. 9–25.

[74] THURSTON, W. P. Three-dimensional manifolds, Kleinian groups and hyperbolic geometry. *Bull. Amer. Math. Soc. (N.S.) 6*, 3 (1982), 357–381.

[75] WALDHAUSEN, F. Heegaard-Zerlegungen der 3-Sphäre. *Topology 7* (1968), 195–203.

[76] WALL, C. T. C. On simply-connected 4-manifolds. *J. London Math. Soc. 39* (1964), 141–149.

[77] WALLACE, A. H. Modifications and cobounding manifolds. *Canad. J. Math. 12* (1960), 503–528.

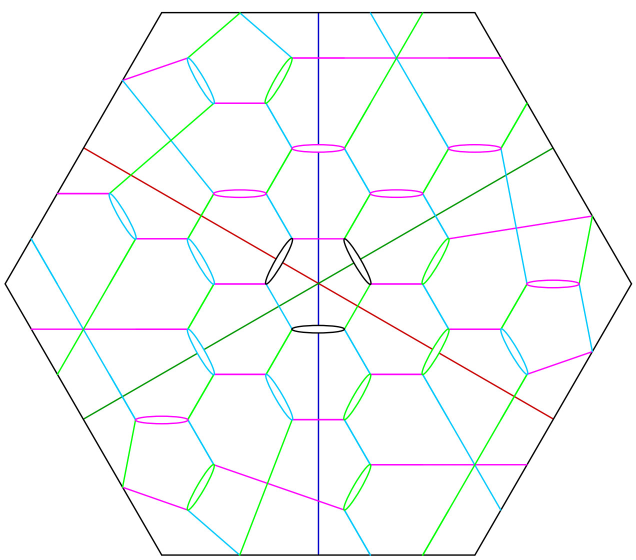
[78] WU, Y.-Q. Dehn surgery on arborescent knots. *J. Differential Geom. 43*, 1 (1996), 171–197.

[79] WU, Y.-Q. Persistently laminar branched surfaces. *Comm. Anal. Geom. 20*, 2 (2012), 397–434.

[80] WU, Y.-Q. Immersed surfaces and Seifert fibered surgery on Montesinos knots. *Trans. Amer. Math. Soc. 365*, 5 (2013), 2469–2488.

[81] WU, Z. A cabling formula for the ν^+ invariant. *Proc. Amer. Math. Soc. 144*, 9 (2016), 4089–4098.

[82] ZUFELT, N. Divisibility of great webs and reducible Dehn surgery. arXiv:1410.3442, 2014.



A genus 22 trisection diagram for the Kummer quartic surface, K3. Each hexagon represents a 21-times-punctured torus once opposite edges are identified. Each elliptical puncture of the left torus is identified with the corresponding elliptical puncture or the right torus via a reflection across the major axis of the ellipse to create a closed, genus 22 surface.