Fibered Ribbon Disks

Jeffrey Meier

Department of Mathematics
Indiana University

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Joint with Kyle Larson (The University of Texas at Austin)

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Disk-knots

Let (B^4, D) be a disk-knot; D a properly embedded disk.

- $(S^3, K) = \partial(B^4, D)$ is a slice knot.
- The double $(S^4, \mathcal{S}) = \mathcal{D}(B^4, D)$ is a 2–knot.

Let E_D, E_K, and $E_{\mathcal{S}}$ denote the exteriors of these objects.

A disk-knot D is homotopy-ribbon if the exterior E_D can be constructed using only 0, 1, and 2–handles.

- K is homotopy-ribbon if K bounds a homotopy-ribbon disk D.
- $\mathcal{S}$ is homotopy-ribbon if $\mathcal{S}$ bounds a homotopy-ribbon 3–ball.

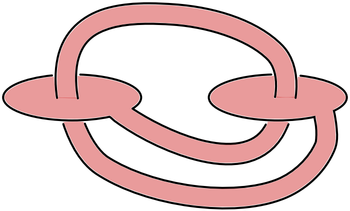
If (B^4, D) is homotopy-ribbon, then $(S^4, \mathcal{S})$ is homotopy-ribbon.

A slice knot K with slice disk D is weakly homotopy-ribbon if

$$\pi_1(E_K) \twoheadrightarrow \pi_1(E_D).$$

Notions of sliceness

A disk knot (B^4, D) is ribbon if D is isotopic to an immersed disk in S^3 with only ribbon singularities.



- K is ribbon if K bounds a ribbon disk D.
- $\mathcal{S}$ is ribbon if $\mathcal{S}$ bounds a ribbon 3–ball.

ribbon $\Longrightarrow$ homotopy-ribbon $\Longrightarrow$ weakly homotopy-ribbon $\Longrightarrow$ slice

- No converse implications are known for 1–knots.
- All 2–knots are slice (Yajima, ’69), but some are not homotopy-ribbon (Cochran, ’83).

Fibered, homotopy-ribbon 1–knots

A knot K is fibered if $E_K \cong \Sigma_g^\circ \times_\varphi S^1$, with φ trivial on $\partial\Sigma_g^\circ$.

- Note that $S_0^3(K) \cong \Sigma_g \times_{\widehat{\varphi}} S^1$ is a closed surface bundle.
- If K bounds a slice disk D, then $\partial(E_D) \cong S_0^3(K)$.

Theorem (Casson and Gordon, 1983)

Let K be a fibered knot with monodromy $\varphi : \Sigma_g^\circ \to \Sigma_g^\circ$.

1. Then, K is homotopy-ribbon if and only if the closed monodromy $\widehat{\varphi} : \Sigma_g \to \Sigma_g$ extends over a handlebody to $\phi : H_g \to H_g$.
2. Moreover, there exists a homotopy-ribbon disk (B, D'), with B a homotopy 4–ball, such that $E_{D'} \cong H_g \times_\phi S^1$

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- It is not known whether $E_D \cong H_g \times_\phi S^1$.

Question: Is B diffeomorphic to B^4?

Question: Is the extension of the fibration unique?

Fibered, homotopy-ribbon 2–knots

A 2–knot $\mathcal{S}$ is fibered if $E_{\mathcal{S}} \cong M^\circ \times_\Phi S^1$, with Φ trivial on ∂M°.

Theorem (Zeeman, 1965)

Let M be the k–fold branched cyclic cover of a knot K. Then M° is the fiber of a fibered 2–knot.

- If $K \subset S^3$ is fibered, then the spin $\mathcal{S}_0(K) \subset S^4$ is a fibered, ribbon 2–knot.

Theorem (Cochran, 1983)

Let $\mathcal{S}$ be a fibered 2–knot with fiber M°. Then, $\mathcal{S}$ is homotopy-ribbon if and only if $M \cong \#_g S^1 \times S^2$.

Corollary: Not all 2–knots are homotopy-ribbon.

FIBERED, HOMOTOPY-RIBBON DISK-KNOTS

A disk-knot D is fibered if $E_D \cong H \times_\phi S^1$, with $\phi : H \to H$, $\partial H \cong \Sigma_g$.

- $K = \partial D$ is fibered knot with monodromy $\varphi : \Sigma_g^\circ \to \Sigma_g^\circ$.
- ∂E_D is the closed surface bundle $S_0^3(K)$ with monodromy $\widehat{\varphi} : \Sigma_g^\circ \to \Sigma_g^\circ$.

THEOREM A (LARSON-M.)

Let D be a fibered disk-knot with fiber H. Then D is homotopy-ribbon if and only if $H \cong H_g$.

Lemma: Any handlebody bundle can be built using only 0, 1, and 2–handles.

- $\mathcal{S} = \mathcal{D}D$ is a fibered, homotopy-ribbon 2–knot with fiber $M = \mathcal{D}(H)$, $\implies M \cong M_h = \#_h S^1 \times S^2$.
- $F = \partial H$ compresses symmetrically and completely in M_h, $\implies H \cong H_g \# M_{\frac{h-g}{2}}$.
- Let $K = \partial D$. Then, $\pi_1(E_K) \twoheadrightarrow \pi_1(E_D)$, and $\pi_1(F) \twoheadrightarrow \pi_1(H)$.
- However, the surjection implies that $h - g = 0$.

2–handle surgery

Suppose that $X = X_0 \cup_f h$ for some 2–handle h with framing curve f. We say that $X' = X_0 \cup_{f'} h$ is obtained by 2–handle surgery on h with slope m if $f \cdot f' = m$.

- Let D be a fibered, homotopy-ribbon disk-knot with boundary K.
- $B^4 = E_D \cup_{f_0} h$, where $h = \nu(D)$. Let $W_m = E_D \cup_{f_m} h$.

$$(B^4, D) \xrightarrow{\text{remove 2–handle}} E_D \xrightarrow{\text{re-attach 2–handle}} (W_m, D)$$

$$(S^3, K) \xrightarrow{\text{0–surgery on } K} S^3_0(K) \xrightarrow{m\text{–surgery on } K'} (Y_m, K)$$

- $Y_m \cong S^3_{-1/m}(K)$, so the knots (Y_m, K) are
  - fibered in (often distinct) $\mathbb{Z}$–homology 3–spheres,
  - homotopy-ribbon in (correspondingly distinct) contractible 4–manifolds,
  - but share a common knot exterior.

2–HANDLE SURGERY

- Suppose that $X = X_0 \cup_f h$, and $X' = X_0 \cup_{f'} h$ is obtained by 2–handle surgery.

Lemma: $\pi_1(X) \cong \pi_1(X')$ and $H_*(X) \cong H_*(X')$.

- Note that the homotopy type of X can change.

Lemma: If the cocore E of h is unknotted in X, then $X' \cong X$.

- Let $E \subset H$ be a nontrivial, properly embedded disk. (Perhaps $H \cong H_g$.)
- Let $\tau_E : H \to H$ denote the diffeomorphism given by a right twist on E.

PROPOSITION

Let $X = H \times_\phi S^1$, and let $X' = H \times_{\phi \circ \tau_E} S^1$. Then X' is obtained from X by 2–handle surgery on a handle h, where the cocore of h is E.

Producing new fibered disk-knots

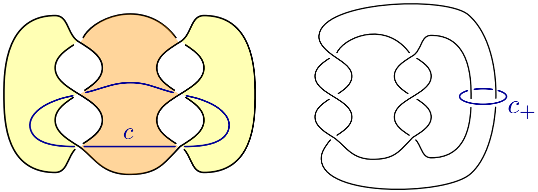
Theorem B (Larson-M.)

Let (B^4, D_0) be a fibered, homotopy-ribbon disk-knot with fiber H and monodromy ϕ, and let $E \subset H$ be a properly embedded, nontrivial disk that is unknotted in B^4. Then, twisting m times along E gives a new fibered, homotopy-ribbon disk-knot (B^4, D_m) with monodromy $\phi \circ \tau_E^m$. Furthermore,

1. The boundary fibered knots (S^3, K_m) have monodromies $\varphi_m = \varphi_0 \circ \tau_c^m$, where c is a Stallings curve for K_0;
2. The collection $\{(B^4, D_m)\}_{m \in \mathbb{Z}}$ contains infinitely many distinct disk-knots;
3. The collection $\{(S^4, \mathcal{S}_m)\}_{m \in \mathbb{Z}}$ contains at most two distinct fibered 2–knots.
4. The exteriors E_{D_m} are homotopy equivalent.

- If D_0 is ribbon, then the D_m may be counterexamples to the Slice Ribbon Conjecture.

SYMMETRIC EQUATORS



A 1–knot K is a symmetric equator for a 2–knot $\mathcal{S}$ if $\mathcal{S}$ is the double of a disk knot along K.

COROLLARY

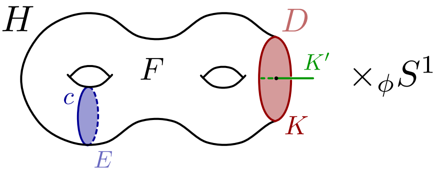
1. Every nontrivial spun 2–knot is the double of infinitely many distinct fibered, ribbon disk knots.
2. Every nontrivial spun 2–knot has infinitely many distinct fibered, ribbon knots (S^3, K) as symmetric equators.

Halving Theorem

Theorem C (Larson-M.)

Let $(S^4, \mathcal{S})$ be a fibered, homotopy-ribbon 2–knot with fiber M_g°. Then, $(S^4, \mathcal{S})$ can be expressed as the double of infinitely many pairs (W_m, D_m), where

1. W_m is a contractible 4–manifold;
2. (W_m, D_m) is a fibered, homotopy-ribbon disk-knot;
3. The boundaries knots (Y_m, K_m) share a common exterior;
4. Infinitely many of the (Y_m, K_m), and therefore the corresponding (W_m, D_m), are distinct.



Symmetric equators

Corollary

1. Every nontrivial, fibered, homotopy-ribbon 2–knot is the double of infinitely many distant fibered, homotopy-ribbon disk knots.
2. Every nontrivial, fibered, homotopy-ribbon 2–knot has infinitely many distinct symmetric equators.

Question: Does every fibered, ribbon 2–knot $(S^4, \mathcal{S})$ have a fibered, ribbon knot (S^3, K) as a symmetric equator?